The Role of Ricci Solitons in the Geometry of (ϵ)-Kenmotsu Manifolds

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ABSTRACT
The object of the present paper is to study Ricci solitons in (ϵ)-Kenmotsu manifolds satisfying S(X,ξ).R=0,R(ξ,X).P =0, P(ξ,X).R=0, R(ξ, X).$\bar{P}$ =0, and $\bar{P}(\xi, X)$.S=0, where P is projective curvature tensor and $\bar{P}$ is pseudo-projective curvature tensor.

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INTRODUCTION
In the differential geometry, the Ricci flow is an intrinsic geometric flow, which was introduced by R. Hamilton ([7],[8]).The Ricci flow is a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat, smoothing our irregularities in the metric. The Ricci flow equation is the evolution equation

$$\frac{d}{dt} g_{ij}(t) = -2 R_{ij}$$

for a Riemannian metric g_{ij}, where R_{ij} is the Ricci curvature tensor. Hamilton ([7]) showed that there is a unique solution to this equation for an arbitrary smooth metric g_{ij}on a closed manifold over a sufficient short time. Hamilton ([7],[8]) also showed that Ricci flow preserves positivity of Ricci curvature tensor in three dimensions and the curvature operator in all dimensions. Ricci solitons are Ricci flows that may change their size but not their shape up to diffeomorphisms. A significant 2-dimensional example of Ricci soliton is the cigar solution[5] which is given by the metric $(dx^2 + dy^2)/(e^{4t} + x^2 + y^2)$ on the Euclidean plane. Although this metric shrinks under the Ricci flow, its geometry remains the same. Such a solution are called steady Ricci solitons.

A Ricci soliton is a triple (g, v, λ) with g a Riemannian metric,V a vector field and λ a real scalar such that

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \quad (1.1)$$

where S is a Ricci tensor of M^n and $\mathcal{L}_V$ denote the Lie –derivative along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as real scalar λ is negative, zero and positive respectively [4].Ricci solitons were studied by several authors in contact and Lorentzian manifold such as Sharma[12],Tripathi [14],Bagewadi and Ingalahalli [1] Wong, De and Liu [15] Nagaraja and Premalatha[10],Bagewadi et all ([2][10]), and others. On the other hand, the study of manifolds with indefinite metrics is of interest from the stand point of physics and relativity. Manifolds with indefinite metrics have been studied by several authors. In 1993,Bejancu and Duggal [3] introduced the concept of (ϵ)-Sasakian manifolds and Xufeng and Xiaoli [13] established that these manifolds are real hyper- surfaces of indefinite Kahlerian manifolds. De and Sarkar[6]introduced (ϵ)-Kenmotsumanifolds and studied some curvature conditions on it. Singh, Pandey, Pandey and Tiwari [13] established the relation between semi-symmetric metric connection and Riemannian connection on (ϵ)-Kenmotsu manifolds and have studied several curvature conditions.

Motivated by these studies, we study Ricci solitons in (ϵ)-Kenmotsu manifolds. In this paper, we have studied Ricci solitons in (ϵ)-Kenmotsu manifolds satisfying S(X,ξ).R=0,R(ξ,X).P =0, P(ξ,X).R=0, R(ξ, X).$\bar{P}$ =0 and $\bar{P}(\xi, X)$.S=0, where P is Projective curvature tensor and $\bar{P}$ is Pseudo-projective curvature tensor of M.

PRELIMINARIES
An n-dimensional smooth manifold (M^n, g) is called an (ϵ)-almost contact metric manifold if

$$\phi^2 X = -X + \eta(X)\xi, \quad (2.1)$$

$$\eta(\xi) = 1, \quad (2.2)$$

$$\epsilon g(X, \xi) = \eta(X) \quad (2.3)$$

$$\epsilon = g(\xi, \xi) \quad (2.4)$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \quad (2.5)$$

where ϵ is 1 or -1 $according$ as ξ is $space-like$ or $time-$ $like$ $vector$ $field$ and $rank$ ϕ is $n-1$.It is important to mention that in the above definition ξ is $never$ a $light-like$ $vector$ $field$. If

$d\eta(X,Y) = g(X,\phi Y)$ (2.6)

for every $X,Y\epsilon TM^n$, then we say that M^n is an (ϵ)-contact metric manifold.
Also,

$$\phi\xi = 0 \text{ and } \eta o\phi = 0. \quad (2.7)$$

If an (ϵ) − contact metric manifold satisfies
$(\nabla_X\phi)(Y) = -g(X,\phi Y)\xi - \epsilon\eta(Y)\phi X$,(2.8)

where ∇ denotes the Riemannian connection of g, then M^n is called an (ϵ) − Kenmotsu manifold [6]. An (ϵ)-almost contact metric manifold is an (ϵ)-Kenmotsu manifold if

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi).(2.9)$$

In an (ϵ) − Kenmotsu manifold, the following relations hold [12]

$$(\nabla_X\eta)(Y)=g(X,Y)-\epsilon\eta(X)\eta(Y),(2.10)$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \quad (2.11)$$

$$R(\xi,X)Y = \eta(Y)X - \epsilon g(X,Y)\xi, \quad (2.12)$$

$$R(X,Y)\phi Z = \phi R(X,Y)Z + \epsilon\{g(Y,Z)\phi X - g(X,Z)\phi Y + g(X,\phi Z)Y - g(Y,\phi Z)X\}, \quad (2.13)$$

$$\eta(R(X,Y)Z) = \epsilon[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)], \quad (2.14)$$

Let (g,V,λ) be a Ricci solitons in an (ϵ)-Kenmotsu manifold. from equation(2.9),we have
$(L_\xi g)(X,Y) = -2[\epsilon g(X,Y) - \eta(X)\eta(Y)]$. (2.15)

In view of equations (1.1) and (2.15),we have

$$S(X,Y) = (\epsilon - \lambda)g(X,Y) - \eta(X)\eta(Y). \quad (2.16)$$

The above equation yields that

$$QX = (\epsilon - \lambda)X - \epsilon\eta(X)\xi, \quad (2.17)$$

$$S(X,\xi)= -\lambda\, g(X, \xi),(2.18)$$

$$r = n\,(\epsilon - \lambda) - \epsilon. \quad (2.19)$$

The projective curvature tensor P is defined as
$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY]$. (2.20)

By virtue of equations (2.14) and (2.17),the projective curvature tensor on (ϵ)-Kenmotsu manifold takes the form
$P(X,Y)Z = [\frac{(n-1)\epsilon-\lambda}{(n-1)}][g(X,Z)Y - g(Y,Z)X] - \frac{\epsilon}{n-1}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi]$,(2.21)

which gives
$\eta(P(X,Y)Z) = [\frac{n\epsilon-2\epsilon-\lambda}{(n-1)}][g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]$. (2.22)

Putting X=ξ in equation (2.21)and using equation (2.2), we get
$P(\xi,Y)Z = [\frac{n-\lambda\epsilon}{(n-1)}][\eta(Z)Y - \epsilon g(Y,Z)\xi]$. (2.23)

Again on putting Z=ξ in equation (2.21)and by the use of equation (2.2),(2.3)and (2.11),we obtain
$P(X,Y)\xi = [\frac{1-\lambda\epsilon}{(n-1)}][\eta(X)Y - \eta(Y)X]$. (2.24)

The pseudo projective curvature tensor $\bar{P}$ [11] is defined as

$$\bar{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y,Z)X - g(X,Z)Y]. \quad (2.25)$$

Putting X=ξ in above equation and using equations (2.12),(2.16)and (2.18),we get

$$\bar{P}(\xi,Y)Z = [a\epsilon + \frac{r\epsilon}{n}\left(\frac{a}{n-1} + b\right)][\epsilon\eta(W)Z + g(Y,Z)\xi] + b[(\epsilon - \lambda)g(Z,W)\xi - \eta(Z)\eta(W)\xi + \lambda\epsilon\eta(W)Z]. \quad (2.26)$$

Also by virtue of equation (2.25),we obtain

$$\bar{P}(X,Y)\xi = [a - b\lambda\epsilon - \frac{r}{n}\left(\frac{a}{n-1} + b\right)][\epsilon g(X,Z)\xi - \eta(Z)X]. \quad (2.27)$$

Now, using equation (2.25)and (2.16),we get

$$\eta(\bar{P}(X,Y)Z) = [a\epsilon + \frac{r}{n}\left(\frac{a}{n-1} + b\right) - b(\epsilon - \lambda)][g(Y,W)\eta(Z) - g(Z,W)\eta(Y)]. \quad (2.28)$$

3. RICCI SOLITONS in (ϵ) − Kenmotsu Manifolds SATISFYING S(ξ,X).R=0

Using the following equations

$$S(X,\xi).R)(U,V)W = ((X\wedge_S\xi).R)(U,V)W = (X\wedge_s\xi)R(U,V)W + R((X\wedge_s\xi)(U,V)W) + R(U,(X\wedge_s\xi)V)W) + R(U,V)(X\wedge_s\xi)W, \quad (3.1)$$

where the endomorphism $(X\wedge_S Y)$ $is defined by$

$$(X\wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y. \quad (3.2)$$

Now, from equations (3.1) and (3.2), we have

$$(S(X,\xi).R)(U,V)W = S(\xi,R(U,V)W)X - S(X,R(U,V)W)\xi + S(\xi,U)R(X,V)W) - S(X,U)R(\xi,V)W + S(\xi,V)R(U,X)W - S(X,V)R(U,\xi)W + S(\xi,W)R(U,V)X - S(X,W)R(U,V)\xi. \quad (3.3)$$

Assuming $(S(X,\xi).R)(U,V)W = 0$, $t$hen above equation reduces to

$$S(\xi,R(U,V)W)X - S(X,R(U,V)W)\xi + S(\xi,U)R(X,V)W - S(X,U)R(\xi,V)W + S(\xi,V)R(U,X)W - S(X,V)R(U,\xi)W + S(\xi,W)R(U,V)X - S(X,W)R(U,V)\xi = 0. \quad (3.4)$$

Taking the inner product of above equation with ξ and using (2.3), (2.4), (2.14), (2.16) and(2.18), we get

$$S(X,R(U,V)W) = -[2\lambda\{g(U,W)\eta(V) - g(V,W)\eta(U)\}\eta(X) + \{g(U,X)\eta(V) - g(V,X)\eta(U)\}\eta(W) + \{S(X,U)\eta(V) - S(X,V)\eta(U)\}\eta(W) + \epsilon\{S(X,V)g(U,W) - S(X,U)g(V,W)\}]. \quad (3.5)$$

which by virtue of equation (2.16),gives

$$(\epsilon - \lambda)g(X,R(U,V,W) + (2\lambda - 1)\{g(U,W)\eta(V) - g(V,W)\eta(U)\}\eta(X) + \{g(U,X)\eta(V) - g(V,X)\eta(U)\}\eta(W) + \{S(X,U)\eta(V) - S(X,V)\eta(U)\}\eta(W) + \epsilon\{S(X,V)g(U,W) - S(X,U)g(V,W)\} = 0. \quad (3.6)$$

Putting X=V=ξ in above equation and using equations (2.2), (2.3)and (2.18), we obtain

$$2\lambda(1-\epsilon)[g(U,W) - \eta(U)\eta(W)] = 0, \quad (3.7)$$

which on contraction gives

$$\lambda(n-1)(1-\epsilon) = 0, \quad (3.8)$$

which gives $\lambda = 0$, for time − like (ϵ) − Kenmotsu manifold.
Thus we can state as follows.

Theorem (1): Ricci Solitons in time like (ϵ)-Kenmotsu manifold satisfying $S(\xi,X).R = 0$, is steady.

RICCI SOLITONS IN $(\epsilon)-$Kenmotsu Manifolds SATISFYING R(ξ, X).P =0.

Let us suppose R(ξ, X).P =0, which gives

(R(ξ, X).P)(Y,Z)W=0

which gives

$$R(\xi,X)P(Y,Z)W-P(R(\xi,X)Y,Z)W-P(Y,R(\xi,X)Z)W-P(Y,Z)R(\xi,X)W=0. \quad (4.1)$$

In view of equation (2.12),we get

$$\eta(P(Y,Z)W)X-\epsilon g(X,P(Y,Z)W)\xi-\eta(Y)P(X,Z)W+\epsilon g(X,Y)P(\xi,Z)W-\eta(Z)P(Y,X)W$$
$$+\epsilon g(X,Z)P(Y,\xi)W-\eta(W)P(Y,Z)X+\epsilon g(X,W)P(Y,Z)\xi=0. \quad (4.2)$$

Now, taking the inner product of above equation with ξ, *we obtain*

$$\eta(P(Y,Z)W)\eta(X)-\epsilon g(X,P(Y,Z)W)-\eta(Y)\eta(P(X,Z)W)+\epsilon g(X,Y)\eta(P(\xi,Z)W)$$
$$-\eta(Z)\eta(P(Y,X)W)+\epsilon g(X,Z)\eta(P(Y,\xi)W)-\eta(W)\eta(P(Y,Z)X)$$
$$+\epsilon g(X,W)\eta(P(Y,Z)\xi)=0. \quad (4.3)$$

By virtue of equation (2.22),above equation takes the form

$$\epsilon g(X,P(Y,Z)W)=[\frac{n\epsilon-2\epsilon-\lambda}{(n-1)}][\epsilon\{g(X,Z)g(Y,W)-g(X,Y)g(Z,W)\}$$
$$+\eta(W)\{g(X,Y)\eta(Z)-g(Y,Z)\eta(X)\}]. \quad (4.4)$$

In view of equation (2.21) above equation reduces to

$$\left[\frac{n\epsilon-2\epsilon-\lambda}{(n-1)}\right][g(Y,W)g(X,Z)-g(Z,W)g(X,Y)]-\frac{1}{n-1}[g(Y,W)\eta(Z)-g(Z,W)\eta(Y)]\eta(X)$$
$$=[\frac{n\epsilon-2\epsilon-\lambda}{(n-1)}][\epsilon\{g(X,Z)g(Y,W)-g(X,Y)g(Z,W)\}$$
$$+\eta(W)\{g(X,Y)\eta(Z)-g(Y,Z)\eta(X)\}] \quad (4.5)$$

Let $\{e_i\}_{i=1}^{n}$ be an orthonormal basis of the tangent space at each point of the manifold.

As putting X=W=e_i and summation over i,1 $\leq$ i $\leq$n, we get

$$\left[\frac{n\epsilon-2\epsilon-\lambda}{(n-1)}\right][g(Y,Z)-\eta(Y)\eta(Z)]=0, \quad (4.6)$$

which gives $\lambda=(n-2)\epsilon$ *because* $g(Y,Z)\neq\eta(Y)\eta(Z)$ (4.7)

Now, suppose ξ is space − like vector field in (ϵ) − Kenmotsu manifolds,then
from equation (4.7),we obtain

$$\lambda=(n-2)>0,$$

which shows that λ is expanding. Thus we can state as follows.

Theorem (2): Ricci Solitons in (ϵ)-Kenmotsu manifold with ξ *as space − like vector field* satisfying R(ξ, X).P $=0$, is expanding.

Again if we assume vector field
ξ as time − like in (ϵ) − Kenmotsu manifolds, then in view of equation (4.7), we obtain

$$\lambda=-(n-2)<0,$$

which shows that λ *is shrinking*.Thus we can state as follows.

Theorem (3): Ricci Solitons in (ϵ)-Kenmotsu manifold admitting ξ *as time − like vector field* satisfying R(ξ, X).P =0, is shrinking.

RICCI SOLITONS IN (ϵ)-KENMOTSU MANIFOLD SATISFYING P(ξ, X).S=0.

The condition P(ξ, X).S=0. implies that

S(P(ξ, X)Y,Z)+S(Y,P(ξ, X)Z)=0. (5.1)

By virtue of equation (2.23) above equation takes the form

$$[\frac{n-\lambda\epsilon}{n-1}][\eta(Y)S(X,Z)-\epsilon g(X,Y)S(Z,\xi)+\eta(Z)S(Y,X)-\epsilon g(X,Z)S(Y,\xi)]=0. \quad (5.2)$$

where K=$[\frac{n-\lambda\epsilon}{n-1}]$

in view of equation (2.16) and (2.18) above equation takes the form

$$[\frac{n-\lambda\epsilon}{n-1}][\epsilon\{g(X,Z)\eta(Y)-g(X,Y)\eta(Z)\}+2\{\lambda g(X,Y)-\eta(X)\eta(Y)\}\eta(Z)]=0 \qquad (5.3)$$

Now, putting equation X=Z=ξ *in above equation and by use of equations* (2.2), (2.3)*and* (2.4), *we obtain*

$$2(\lambda\epsilon-1)\eta(Y)=0,$$

$$2(\lambda\epsilon-1)=0,\ \ \{where\ \ \eta(Y)\neq 0.\}$$

$$\lambda=\frac{1}{\epsilon} \qquad (5.4)$$

Now, suppose ξ is space − like vector field in (ϵ) − Kenmotsu manifolds, then from equation (5.4),we obtain

λ= 1 > 0

which shows thatλ is expanding.Thus we can state as follows.

Theorem (4): Ricci Solitons in (ϵ)-Kenmotsu manifold with ξ *as space − like vector field* satisfying P(ξ, X).S= 0, is expanding.

Again if we assume vector field ξ as time − like in (ϵ) − Kenmotsu manifolds, then in view of equation (5.4), we obtain

λ= −1 < 0

which shows thatλ is shrinking. Thus we can state as follows.

Theorem (5): Ricci Solitons in (ϵ)-Kenmotsu manifold admitting ξ *as time − like vector field* satisfying P(ξ, X).S=0, is shrinking.

RICCI SOLITONS IN (ϵ) − *Kenmotsu* Manifolds SATISFYING R(ξ, X).$\bar{P}$=0.

Let R(ξ, X).$\bar{P}$ =0, then we have

$$R(\xi,X)\bar{P}(Y,Z)W-\bar{P}(R(\xi,X)Y,Z)W-\bar{P}(Y,R(\xi,X)Z)W-\bar{P}(Y,Z)R(\xi,X)W=0. \qquad (6.1)$$

By virtue of equation (2.12) above equation reduces to

$$\eta(\bar{P}(Y,Z)W)X-\epsilon g(X,\bar{P}(Y,Z)W)\xi-\eta(Y)\bar{P}(X,Z)W+\epsilon g(X,Y)\bar{P}(\xi,Z)W$$
$$-\eta(Z)\bar{P}(Y,X)W+\epsilon g(X,Z)\bar{P}(Y,\xi)W)-\eta(W)\bar{P}(Y,Z)X+\epsilon g(X,W)\bar{P}(Y,Z)\xi=0. \qquad (6.2)$$

Taking the inner product of above equation with ξ and using equation (2.2)and (2.3), we get

$$\eta(\bar{P}(Y,Z)W)\eta(X)-\epsilon g(X,\bar{P}(Y,Z)W)\xi-\eta(Y)\eta(\bar{P}(X,Z)W))+\epsilon g(X,Y)\eta(\bar{P}(\xi,Z)W)$$
$$-\eta(Z)\eta(\bar{P}(Y,X)W)+\epsilon g(X,Z)\eta(\bar{P}(Y,\xi)W)-\eta(W)\eta(\bar{P}(Y,Z)X)+\epsilon g(X,W)\eta(\bar{P}(Y,Z)\xi)=0, \qquad (6.3)$$

Using equations (2.25)and (2.28) in above equation, we obtain

$$K[g(X,Z)g(Y,W)-g(X,Y)g(Z,W)]=\epsilon g(X,\bar{P}(Y,Z)W) \qquad (6.4)$$

Where K=$[a\epsilon+\frac{r}{n}(\frac{a}{n-1}+b)-b(\epsilon-\lambda)]$

In view of equations (2.16) and (2.25) above equation reduces to

$$K[g(X,Z)g(Y,W)-g(X,Y)g(Z,W)]=\epsilon g(X,(R(Y,Z)W)+b[S(Z,W)g(X,Y)$$
$$-S(Y,W)g(X,Z)]-\frac{r}{n}[\frac{a}{n-1}+b][g(Z,W)g(Y,W)g(X,Z)] \qquad (6.5)$$

Let $\{e_i\}_{i=1}^{n}$ be an orthonormal basis of the tangent space at each point of the manifold.

As putting X=W=e_i and summation over i,1 ≤ i ≤n, we get

$$aS(Y,Z)=0 \qquad (6.6)$$

Using equation (2.16),we get

$$(\epsilon-\lambda)g(Y,Z)-\eta(Y)\eta(Z)=0,$$

which on substituting Y=Z=ξ *gives*

$$\lambda=\epsilon.$$

which shows that λ is expanding.Thus we can state as follows.

Theorem (6): Ricci Solitons in (ϵ)-Kenmotsu manifold satisfying $R(\xi, X).\overline{P} = 0$, is expanding or shrinking according ξ is space − like vector field or time − likevector field respectiverly.

RICCI SOLITONS IN (ϵ) − Kenmotsu Manifolds SATISFYING $\overline{P}(\xi, X).S = 0$.

The condition $\overline{P}(\xi, X). S = 0$, which implies

$$S(\bar{P}(\xi,X)Y,Z)+S(Y,\bar{P}(\xi,X)Z)=0, \quad (7.1)$$

Using equation (2.26) in above equation, we get

$$K[\eta(Y)S(X,Z) + \eta(Z)S(Y,X) + \lambda g(X,Y)\eta(Z) + \lambda g(X,Z)\eta(Y) - (\lambda\epsilon)[(\epsilon-\lambda)g(X,Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z) - \eta(Y)S(X,Z) + (\epsilon-\lambda)g(X,Z)\eta(Y) - \eta(Z)S(Y,X)] = 0 \quad (7.2)$$

In view of equation (2.16), above equation reduces

$$\left[a\epsilon + \frac{r\epsilon}{n}\left(\frac{a}{n-1} + b\right)\right][\epsilon\{g(X,Z)\eta(Y) + g(Y,X)\eta(Z)\} + 2\eta(X)\eta(Y)\eta(Z)] = 0 \quad (7.3)$$

Let $\{e_i\}_{i=1}^{n}$ be an orthonormal basis of the tangent space at each point of the manifold.
As putting X=Z=e_i and summation over i,$1 \le i \le$n, we get

$$\left[a\epsilon + \frac{r\epsilon}{n}\left(\frac{a}{n-1} + b\right)\right](n\epsilon - 1)\eta(Y) = 0, \quad (7.4)$$

Now, suppose ξ is space − like vector field in (ϵ) − Kenmotsu manifolds,then from equation (7.5),we obtain

$$\lambda = \frac{a\epsilon(2n+1) + b\epsilon(n-1)(n+1)}{an + bn(n-1)} > 0$$

this shows that λ is expanding.Thus we can state as follows.

Theorem (7): Ricci Solitons in (ϵ)-Kenmotsu manifold with ξ as space − like vector field satisfying condition $\overline{P}(\xi, X). S = 0$, is expanding.

Example: Consider 3-dimensional manifold
M={(x,y,z)ϵR^3;z$\neq$ 0}, where (x, y, z)are the standard coordinates in R^3.
Let $\{e_1, e_2, e\}$ be linearly independent given by

$$e_1 = z\frac{\partial}{\partial x}, e_2 = z\frac{\partial}{\partial y}, e_3 = -z\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by $g(e_1, e_3)=g(e_2, e_3)=g(e_1, e_2)=0$,
$g(e_1, e_1)=g(e_2, e_2)=g(e_3, e_3)=\epsilon$,

where $\epsilon = \pm 1$, Let η *be the* $1-$ *form defined by* $\eta(Z) = \epsilon g(Z, e_3)$ for any Z∈TM^n.
Letϕ *be the* $(1,1)-$ *tensor field defiend by*

$$\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0.$$

Then using the linearity property of ϕ and g we have

$$\eta(e_3) = 1, \phi^2 Z = -Z + \eta(Z)e_3,$$
$$g(\phi Z, \phi W)=g(Z,W)-\eta(Z)\eta(W),$$

for any vector fields U,W∈ TM^n.

Let ∇ be the Levi − Civita connection with respect to metric g . we have

$$[e_1, e_2]=0, \quad [e_1, e_2]=\epsilon e_1, \quad [e_2, e_3] = \epsilon e_2.$$

The Riemannian connection ∇ of the metric g is given by Koszul's formula

$$2g(\nabla_X Y, Z)=X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]),$$

And by virtue of it we have

$$\nabla_{e_1} e_3 = \epsilon e_1, \nabla_{e_2} e_3 = \epsilon e_2, \quad \nabla_{e_3} e_3 = 0,$$
$$\nabla_{e_1} e_3 = 0, \quad \nabla_{e_2} e_2 = -\epsilon e_3, \quad \nabla_{e_3} e_3 = 0,$$
$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_1 = 0, \quad \nabla_{e_3} e_1 = 0,$$

For ξ=e_3. Hence the manifold under consideration is an (ϵ) − Kenmotsu manifolds of Three–dimension

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