Self-Maps and Common Fixed Points in Dislocated Quasi-Metric Spaces

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ABSTRACT

In this paper, a common fixed point theorem of a pair of self-maps is proved by omitting continuity requirement in dislocated quasi metric spaces. It extends and generalizes the result of Sarma et al. [5, Theorem 5] to two self-maps by employing a more generalized contraction. It further unifies the results of Dubey et al. [2, Theorem 3.1 and Theorem 3.2], and some well-known fixed point results in the literature.

KEYWORDS: Complete dislocated quasi-metric, Contraction, Common fixed point.

AMS Subject Classification: 47H10, 54H25.

I. INTRODUCTION

Dislocated topologies serves as an essential tool in view of its utility in the pursuit of developing logic programming (see [3], [4]). In 2000, Hitzler and Seda [4] proved a fixed point theorem in complete dislocated metric spaces as a generalization of the celebrated Banach contraction principle.

In 2006, Zeyada et al. [7] initiated the notion of complete dislocated quasi-metric space as a generalization of dislocated metric space, and generalized the result of Hitzler et al. [4] in such space. In 2008, Aage and Salunke [1] generalized the result of Zeyada et al. [7] by proving a fixed point theorem for Kannan type of contraction in complete dislocated quasi-metric space. Afterwards, a few papers dealt with fixed points in such space were obtained (for instance [5], [6] etc).

In 2014, Sarma et al. [5, Theorem 5] improved the result of Aage and Salunke [1, Theorem 3.3] by omitting continuity requirement, stated below as Theorem 1.1.

Theorem 1.1. Let (X,d) be a complete dq-metric space, and let $T: X \to X$ be a self-map satisfying the following condition:

$$d(Tx,Ty) \le a\{d(x,Tx)+d(y,Ty)\}$$

for all $x, y \in X$, where $0 \le a < \frac{1}{2}$.

Then T has a unique fixed point in X.

The objective of this paper is to extend and generalize the result of Sarma et al. [5, Theorem 5] to two self-maps by employing a more generalized contraction, and then to unify the results of Dubey et al. [2, Theorem 3.1 and Theorem 3.2] and Aage et al. [1, Theorem 3.3].

Throughout this paper, ¥ denotes the set of positive integers and $\text{¥}_0 = \text{¥} \cup \{0\}$.

II. PRELIMINARIES

We need to retrieve the following relevant definitions and results in the sequel.

Definition 2.1. ([7]). Let X be a non-empty set and let $d: X \times X \to [0,\infty)$ be a function satisfying the following conditions:

(i) $d(x,y) = d(y,x) = 0$ implies $x = y$

(ii) $d(x,y) \le d(x,z) + d(z,y)$, for all $x, y, z \in X$.

Then d is called a dislocated quasi-metric (in short, dq-metric) on X, and the pair (X,d) is called a dislocated quasi-metric space (in short, dq-metric space).

In addition, if d satisfies $d(x,y) = d(y,x)$ for all $x, y \in X$, then it is called a dislocated metric.

A metric on a set is an example of dislocated metric which is also a dislocated quasi metric, but a dislocated quasi-metric is not necessarily dislocated metric and so it is not a metric.

A simple illustration of these facts is furnished in the following.

Example 2.2. Let $X=[0,1]$. Define $d: X\times X\rightarrow[0,\infty)$ by $d(x,y)=|x-y|+|x|$ for all $x,y\in X$. Then d is a dislocated quasi-metric space on X, but symmetric condition fails to hold and therefore, it is neither dislocated metric nor metric on X.

In what follows, X denotes dislocated quasi-metric space (X,d).

Definition 2.3. ([7]). A sequence $\{x_n\}$ in dq-metric space X is called dq-convergent if for $n\in$ ¥, $\lim_{n\to\infty} d(x_n,x)=\lim_{n\to\infty} d(x,x_n)=0$.

In this case, x is called a dislocated quasi limit (in short, dq-limit) of the sequence $\{x_n\}$.

Lemma 2.4. ([7]). dq-limits in a dq-metric space are unique.

Lemma 2.5. ([7]). Every subsequence of dq-convergent sequence to a point x_0 is dq-convergent to x_0.

Definition 2.6. ([7]). A sequence $\{x_n\}$ in dq-metric space X is called Cauchy sequence if for each $\varepsilon>0$, there exists $n_0\in$ ¥ such that $d(x_m,x_n)<\varepsilon$ or $d(x_n,x_m)<\varepsilon$ for all $m,n\geq n_0$.

Definition 2.7. ([7]). A dq-metric space X is called complete if every Cauchy sequence in it is dq-convergent.

III. MAIN RESULT

Theorem 3.1. Let (X,d) be a complete dq-metric space, and let $S,T: X\rightarrow X$ be a pair of self-maps satisfying the following condition:

$$d(Sx,Ty)\leq a_1 d(x,y)+a_2\{d(x,Sx)+d(y,Ty)\}+a_3\{d(x,Ty)+d(y,Sx)\} \quad .... \quad (3.1)$$

for all $x,y\in X$, where $a_i\geq 0$ with $a_1+2a_2+4a_3<1$.

Then S and T have a unique common fixed point in X.

Proof. Let us choose $x_0\in X$ arbitrary. We define a sequence $\{x_n\}$ in X such that $x_{2n+1}=Sx_{2n}$ and $x_{2n+2}=Tx_{2n+1}$ for all $n\in$ ¥$_0$.

We consider $d(x_{2n+1},x_{2n+2})=d(Sx_{2n},Tx_{2n+1})$.

In view of (3.1), we have

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &\leq a_1 d(x_{2n},x_{2n+1})+a_2\{d(x_{2n},Sx_{2n})+d(x_{2n+1},Tx_{2n+1})\} \\ &\qquad +a_3\{d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})\} \\ &= a_1 d(x_{2n},x_{2n+1})+a_2\{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})\} \\ &\qquad +a_3\{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})\} \\ &\leq a_1 d(x_{2n},x_{2n+1})+a_2\{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})\} \\ &\qquad +a_3\{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})+d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})\} \\ &= (a_1+a_2+2a_3)\, d(x_{2n},x_{2n+1})+(a_2+2a_3)\, d(x_{2n+1},x_{2n+2}) \end{aligned}$$

$$\Rightarrow \quad (1-a_2-2a_3)\, d(x_{2n+1},x_{2n+2})\leq (a_1+a_2+2a_3)\, d(x_{2n},x_{2n+1})$$

$$\Rightarrow \quad d(x_{2n+1},x_{2n+2})\leq\left(\frac{a_1+a_2+2a_3}{1-a_2-2a_3}\right) d(x_{2n},x_{2n+1})$$

$$\Rightarrow \quad d(x_{2n+1},x_{2n+2})\leq \lambda\, d(x_{2n},x_{2n+1}), \text{ where } \lambda=\frac{a_1+a_2+2a_3}{1-a_2-2a_3}<1.$$

Similarly, we have $d(x_{2n},x_{2n+1})\leq\lambda\, d(x_{2n-1},x_{2n})$.

So, we obtain $d(x_{2n+1},x_{2n+2})\leq\lambda^2 d(x_{2n-1},x_{2n})$.

Proceeding in this way, we have $d(x_{2n+1},x_{2n+2})\leq\lambda^{2n+1} d(x_0,x_1)$.

We claim that $\{x_n\}$ is a Cauchy sequence in X.

Now, for $n,k\in$ ¥, we see that

$$\begin{aligned} d(x_n,x_{n+k}) &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})+ \quad .... \quad +d(x_{n+k-1},x_{n+k}) \\ &\leq(\lambda^n+\lambda^{n+1}+\lambda^{n+2}+ \quad .... \quad +\lambda^{n+k-1})\, d(x_0,x_1) \\ &\leq(\lambda^n+\lambda^{n+1}+\lambda^{n+2}+ \quad .... \quad)\, d(x_0,x_1) \\ &=\left(\frac{\lambda^n}{1-\lambda}\right) d(x_0,x_1). \end{aligned}$$

Since $\lambda < 1$, $\lambda^n \to 0$ as $n \to \infty$ and so, $d(x_n, x_{n+k}) \to 0$. Similarly, we can show that $d(x_{n+k}, x_n) \to 0$. Thus, $\{x_n\}$ is a Cauchy sequence in X. It follows that completeness of X implies existence of $u \in X$ such that $\lim_{n\to\infty} d(x_n, u) = \lim_{n\to\infty} d(u, x_n) = 0$. Also the subsequences $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ of the sequence $\{x_n\}$ converge to u.

Now, we claim that $Su = Tu = u$.

We have $d(u, Su) \le d(u, x_{2n}) + d(x_{2n}, Su)$

$$= d(u, x_{2n}) + d(Tx_{2n-1}, Su)$$

By using (3.1), we have

$$d(u,Su) \le d(u,x_{2n}) + a_1 d(x_{2n-1},u) + a_2\{d(x_{2n-1},Tx_{2n-1}) + d(u,Su)\} + a_3\{d(x_{2n-1},Su) + d(u,Tx_{2n-1})\}$$

$$= d(u,x_{2n}) + a_1 d(x_{2n-1},u) + a_2\{d(x_{2n-1},x_{2n}) + d(u,Su)\} + a_3\{d(x_{2n-1},Su) + d(u,x_{2n})\}$$

Taking limit $n \to \infty$, we get

$$(1 - a_2 - a_3)\, d(u,Su) \le 0,$$

which is possible if $d(u,Su) = 0$, since $(1 - a_2 - a_3) \ne 0$.

Therefore, $d(u,Su) = 0$.

Also, we have $d(Su,u) \le d(Su,x_{2n}) + d(x_{2n},u)$

$$= d(Su,Tx_{2n-1}) + d(x_{2n},u)$$

In view of (3.1), we have

$$d(Su,u) \le a_1 d(u,x_{2n-1}) + a_2\{d(u,Su) + d(x_{2n-1},Tx_{2n-1})\} + a_3\{d(u,Tx_{2n-1}) + d(x_{2n-1},Su)\} + d(x_{2n},u)$$

$$= a_1 d(u,x_{2n-1}) + a_2\{d(u,Su) + d(x_{2n-1},x_{2n})\} + a_3\{d(u,x_{2n}) + d(x_{2n-1},Su)\} + d(x_{2n},u)$$

Taking limit $n \to \infty$, we get

$$d(Su,u) \le (a_2 + a_3)\, d(u,Su).$$

Since $d(u,Su) = 0$, $d(Su,u) \le 0$ and so, $d(Su,u) = 0$.

Therefore, $d(u, Su) = d(Su, u) = 0$ and so, $Su = u$.

Similarly, it can be shown that $Tu = u$.

It follows that $Su = Tu = u$, and therefore, u is a common fixed point of S and T.

We claim that u is the unique common fixed point of S and T.

Since u is a common fixed point of S and T, we have

$$d(u,u) = d(Su,Tu)$$
$$\le a_1 d(u,u) + a_2\{d(u,Su) + d(u,Tu)\} + a_3\{d(u,Tu) + d(u,Su)\}$$
$$= (a_1 + 2a_2 + 2a_3)\, d(u,u)$$

$$\Rightarrow (1 - a_1 - 2a_2 - 2a_3)\, d(u,u) \le 0,$$

which is possible if $d(u,u) = 0$, since $1 - a_1 - 2a_2 - 2a_3 \ne 0$.

Therefore, $d(u,u) = 0$.

If possible, let there be another common fixed point v of S and T.

Then $d(u,v) = d(Su,Tv)$

$$\le a_1 d(u,v) + a_2\{d(u,Su) + d(v,Tv)\} + a_3\{d(u,Tv) + d(v,Su)\}$$
$$= a_1 d(u,v) + a_2\{d(u,u) + d(v,v)\} + a_3\{d(u,v) + d(v,u)\}$$
$$= (a_1 + a_3)\, d(u,v) + a_3 d(v,u) \quad \text{....} \quad (3.2)$$

Similarly, we have $d(v,u) \le (a_1 + a_3)\, d(v,u) + a_3 d(u,v)$ .... (3.3)

From (3.2) and (3.3), we have

$$|d(u,v) - d(v,u)| \le |a_1 + a_3 - a_3|\,|d(u,v) - d(v,u)|$$

which implies $d(u, v) = d(v, u)$, since $0 \le a_1 < 1$.

From (3.2), we get

$d(u, v) \le (a_1 + 2a_3)\, d(u, v)$, which gives $d(u,v) = 0$, since $a_1 + 2a_3 < 1$.

Further, we obtain $d(u, v) = d(v, u) = 0$, which implies $u = v$.

Hence, u is a unique common fixed point of S and T.

This completes the proof.

By setting $S = T$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2.

Let (X, d) be a complete dq-metric space, and let $T : X \to X$ be a self-map satisfying the following condition:

$$d(Tx, Ty) \le a_1 d(x, y) + a_2\{d(x, Tx) + d(y, Ty)\} + a_3\{d(x, Ty) + d(y, Tx)\}$$

for all $x, y \in X$, where $a_i \ge 0$ with $a_1 + 2a_2 + 4a_3 < 1$.

Then T has a unique fixed point in X.

Remark 3.3. If $a_1 = a_3 = 0$ in Corollary 3.2, we obtain Theorem 1.1 (Sarma et al. [5, Theorem 5]) as a corollary of Theorem 3.1.

Taking into account that T is continuous and $S = T$ in the Theorem 3.1, we obtain the following corollary.

Corollary 3.4.

Let (X, d) be a complete dq-metric space, and let $T : X \to X$ be a continuous self-map satisfying the following condition:

$$d(Tx, Ty) \le a_1 d(x, y) + a_2\{d(x, Tx) + d(y, Ty)\} + a_3\{d(x, Ty) + d(y, Tx)\}$$

for all $x, y \in X$, where $a_i \ge 0$ with $a_1 + 2a_2 + 4a_3 < 1$.

Then T has a unique fixed point in X.

Remark 3.5. Corollary 3.4 reduces to Theorem 3.1 of Dubey et al. [2] if we set $a_3 = 0$.

Remark 3.6. Corollary 3.4 reduces to Theorem 3.2 of Dubey et al. [2] if we take $a_2 = 0$.

Remark 3.7. Corollary 3.4 reduces to Theorem 3.3 of Aage et al. [1] by putting $a_1 = 0$ and $a_3 = 0$.

IV. REFERENCES

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