

CUBE DIFFERENCE LABELING OF SOME SPECIAL GRAPH FAMILIES

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ABSTRACT

A new labeling and a new graph called cube difference labeling and the cube difference is defined. Let G be a (p,q) graph. G is said to have a cube difference labeling if there exists injection $f:V(G) \rightarrow \{0,1,2,\dots,p-1\}$ such that the edge set of G has assigned a weight defined by the absolute cube difference of its end vertices, the resulting weights are distinct. A graph which admits cube difference labeling is called cube difference graph. The cube difference labeling for some special graph families like **Pan graph**, **Lollipop graph**, **Barbell graph**, **Sunlet graph**, **Sparkler graph**, **Fan graph**, **Triangular Snake Graph**, **Z-P_n graph** are discussed in this paper.

Keywords: *Cube difference labeling, Cube difference graph.*

I. INTRODUCTION

All graph in this paper are simple finite undirected and nontrivial graph $G = (V,E)$ with vertex set V and the edge set E . A function f is a cube difference labeling of a graph G of size n if f is an injection from $V(G)$ to the set $\{0,1,2,\dots,p-1\}$ such that, when each edge uv of G has assigned the weight $|(f(u))^3 - (f(v))^3|$, the resulting weights are distinct. The notion of square difference labeling was introduced by J. Shima [4]-[6]. Graph labeling can also be applied in areas such as communication network, mobile telecommunications, and medical field. A dynamic survey on graph labeling is regularly updated by Gallian [2] and it is published by Electronic Journal of Combinatory. The notation and terminology used in this paper are taken from [1].

Definition 1.1: Let $G = (V(G),E(G))$ be a graph. G is said to be cube difference labeling if there exist a injection $f:V(G) \rightarrow \{0,1,2,\dots,p-1\}$ such that the induced function $f^*:E(G) \rightarrow N$ given by $f^*(uv) = |(f(u))^3 - (f(v))^3|$ is injection.

Definition 1.2: A graph which satisfies the cube difference labeling is called the cube difference graph.

Definition 1.3: The Pan graph is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge. It is denoted by P_n .

Definition 1.4: The Lollipop graph is the graph obtained by a Complete graph K_m to a path P_n with a bridge. It is denoted by $L_{m,n}$.

Definition 1.5: The Barbell graph is obtained by connecting two copies of K_n by a bridge. It is denoted by B_n .

Definition 1.6: The Sunlet graph S_n is a graph obtained from a cycle C_n attached a pendent edge at each vertex of the n -cycle. It has $2n$ vertices and $2n$ edges.

Definition 1.7: The Sparkler graph P_{m+n} is a graph obtained from a path P_m and appending n edges to an end point. It has $m+n$ vertices and $m+n-1$ edges.

Definition 1.8: A fan graph obtained by joining all the vertices of a path P_n to a further vertex, called the Centre. It is denoted by F_n . It has $n+1$ vertices and $2n-1$ edges.

Definition 1.9: The Triangular Snake T_n is obtained the path P_n by replace each of the path by a triangle. It has $2n+1$ vertices and $3n$ edges.

Definition 1.10: In a pair path P_n , i^{th} vertex of a path P_1 is joined with $i+1^{\text{th}}$ vertex of a path P_2 . It is denoted by $Z-P_n$.

II. MAIN RESULT

Theorem: 2.1

The Pan graph P_n admits a Cube difference labeling.

Proof:

Let P_n be a Pan graph. Let $|V(G)| = n+1$ and $|E(G)| = n+1$.

The mapping $f:V(G) \rightarrow \{0,1,2,\dots,n-1\}$ is defined by

$f(u) = 0$ and $f(u_i) = i+2$, $0 \leq i \leq n-1$ and the induced function, $f^*:E(G) \rightarrow N$ is defined by

and here the edge sets are $E_1 = \{u_i u_{i+1} / 0 \leq i \leq n-1\}$ and $E_2 = \{u u_i / i=1\}$

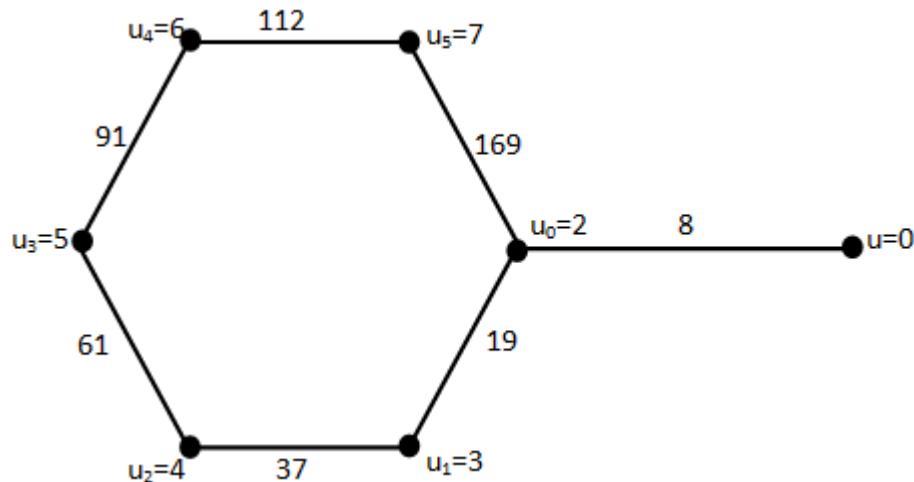
and the edge labeling are,

$$\begin{aligned} (i) \quad f^*(u_i u_{i+1}) &= \bigcup_{i=0}^{n-1} |(f(u_i))^3 - (f(u_{i+1}))^3| \\ &= \bigcup_{i=0}^{n-1} |(i+1)^3 - (i+3)^3| \\ &= \bigcup_{i=0}^{n-1} (3i^2 + 15i + 19) \\ &= \{19, 37, 61, \dots\} \end{aligned}$$

$$\begin{aligned} (ii) \quad f^*(u u_0) &= (i+2)^3, i=0 \\ &= 8. \end{aligned}$$

Here all the edges are distinct. Hence, the Pan graph P_n admits a Cube difference labeling.

Example 2.2: The Pan graph P_6 is a cube difference graph.



Theorem: 2.3

The Lollipop graph $L_{m,n}$ admits a Cube difference labeling.

Proof:

Let $L_{m,n}$ be a Lollipop graph. Let $|V(G)| = m+n$ and $|E(G)| = m+n+2$.

The mapping $f:V(G) \rightarrow \{0,1,2,\dots,n-1\}$ is defined by

$f(u_i) = i$, $0 \leq i \leq n-1$ and $f(v_i) = i+1$, $n-1 \leq i \leq 2(m-1)$ the induced function,

$f^*:E(G) \rightarrow N$ is defined by and here the edge sets are $E_1 = \{u_i u_{i+1} / 0 \leq i \leq n-1\}$ and $E_2 = \{v_i v_{i+1} / n \leq i \leq 2(m-1)\}$,

$E_3 = \{v_i v_{i+2} / i=3\}$ and $E_4 = \{v_{i+2} v_{i+4} / i=2\}$ and the edge labeling are,

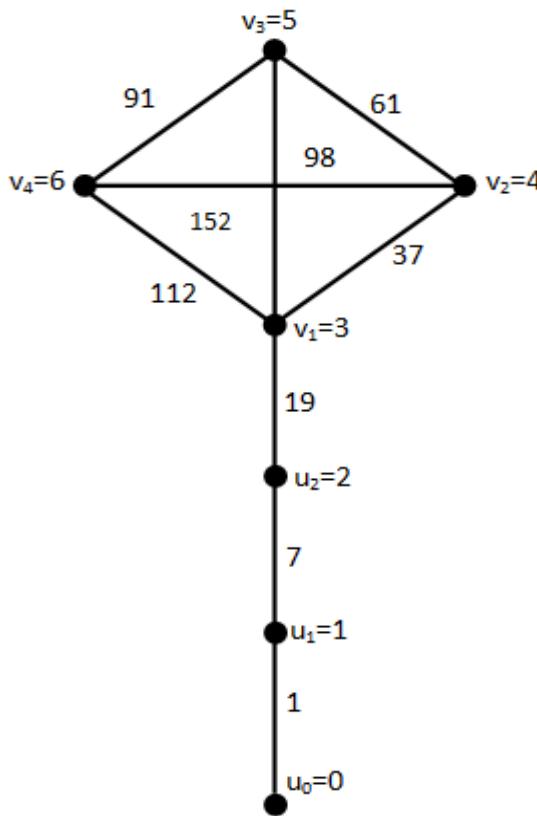
$$\begin{aligned} (i) \quad f^*(u_i u_{i+1}) &= \bigcup_{i=0}^{n-1} |(f(u_i))^3 - (f(u_{i+1}))^3| \\ &= \bigcup_{i=0}^{n-1} |(i)^3 - (i+1)^3| \\ &= \bigcup_{i=0}^{n-1} (3i^2 + 3i + 1) \\ &= \{1, 7\}. \end{aligned}$$

$$\begin{aligned} (ii) \quad f^*(v_i v_{i+1}) &= \bigcup_{i=1}^m |(f(v_i))^3 - (f(v_{i+1}))^3| \\ &= \bigcup_{i=1}^m (3i^2 + 3i + 7). \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad f^*(v_i v_{i+2}) &= |(f(v_i))^3 - (f(v_{i+2}))^3| \\
 &= |(i)^3 - (i+2)^3| \\
 &= 6i^2 + 24i + 26, \quad i=2 \\
 &= 98 \\
 \text{(iv)} \quad f^*(v_{i+1} v_{i+3}) &= |(f(v_{i+1}))^3 - (f(v_{i+3}))^3| \\
 &= |(i+2)^3 - (i+4)^3| \\
 &= 6i^2 + 36i + 56, \\
 &= 152.
 \end{aligned}$$

Here all the edges are distinct. Hence, the Lollipop graph $L_{m,n}$ admits a Cube difference labeling.

Example 2.4: $L_{4,3}$



Theorem: 2.5

The Barbell graph B_n admits a Cube difference labeling.

Proof:

Let B_n be the Barbell graph. Let $|V(G)|=2n$ and $|E(G)|=2n+1$.

The mapping $f:V(G) \rightarrow \{0,1,2,\dots,2n-1\}$ is defined by $f(u_i)=i+1$, $0 \leq i \leq 2n-1$. and induced function $f^*:E(G) \rightarrow N$ is defined by, and here the sets are,

$E_1=\{u_iu_{i+1} / 0 \leq i \leq n-1\}$ and $E_2=\{u_iu_{i+2} / i=1\}$ and $E_3=\{u_{i+2}u_{i+4} / i=2\}$.

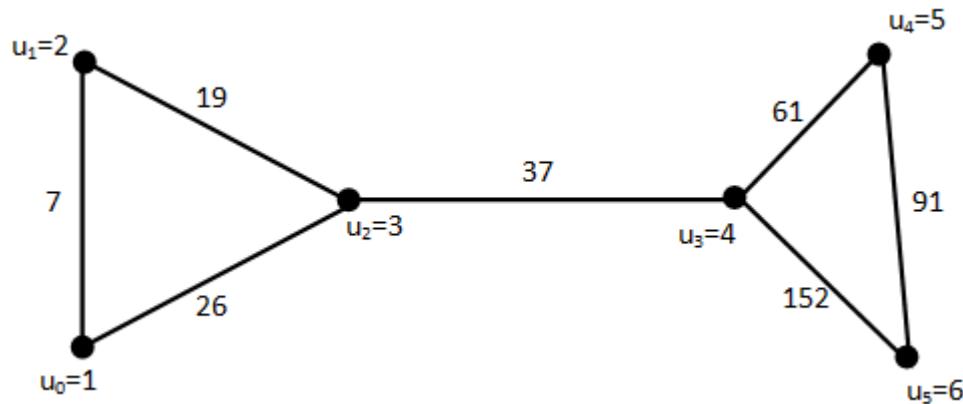
$$\begin{aligned}
 \text{(i)} \quad f^*(u_i u_{i+1}) &= \bigcup_{i=0}^{n-1} |(f(u_i))^3 - (f(u_{i+1}))^3| \\
 &= \bigcup_{i=0}^{n-1} |(i+1)^3 - (i+2)^3| \\
 &= \bigcup_{i=0}^{n-1} (3i^2 + 9i + 7) \\
 &= \{1, 7, 19, 37, \dots, 91\}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad f^*(u_i u_{i+2}) &= |i^3 - (i+2)^3| \\
 &= 6i^2 + 12i + 8, \quad i=1 \\
 &= 26
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad f^*(u_{i+2}u_{i+4}) &= |(f(u_{i+2}))^3 - (f(u_{i+4}))^3| \\
 &= |(i+2)^3 - (i+2)^3| \\
 &= 6i^2 + 36i + 56 \quad , i=2 \\
 &= 152.
 \end{aligned}$$

Hence all the edges are distinct. Hence the graph B_n admits a Cube difference labeling.

Example 2.6: The Barbell graph B_3 is a Cube difference graph



Theorem: 2.7

The Sunlet graph S_n admits a Cube difference labeling.

Proof:

Let S_n be a Sunlet graph. Let $|V(G)|=2n$ and $|E(G)|=2n$.

The mapping $f: V(G) \rightarrow \{0, 1, 2, \dots, 2n-1\}$ is defined by $f(u_i) = i$, $0 \leq i \leq 2n-1$

and the induced function $f^*: E(G) \rightarrow N$ is defined by, and here the sets are,

$E_1 = \{u_iu_{i+1} / 0 \leq i \leq n-1\}$ and $E_2 = \{u_{n-1}u_0\}$

$E_3 = \{u_iu_{n+i} / 0 \leq n+i \leq 2n-1\}$ and the edge labeling are,

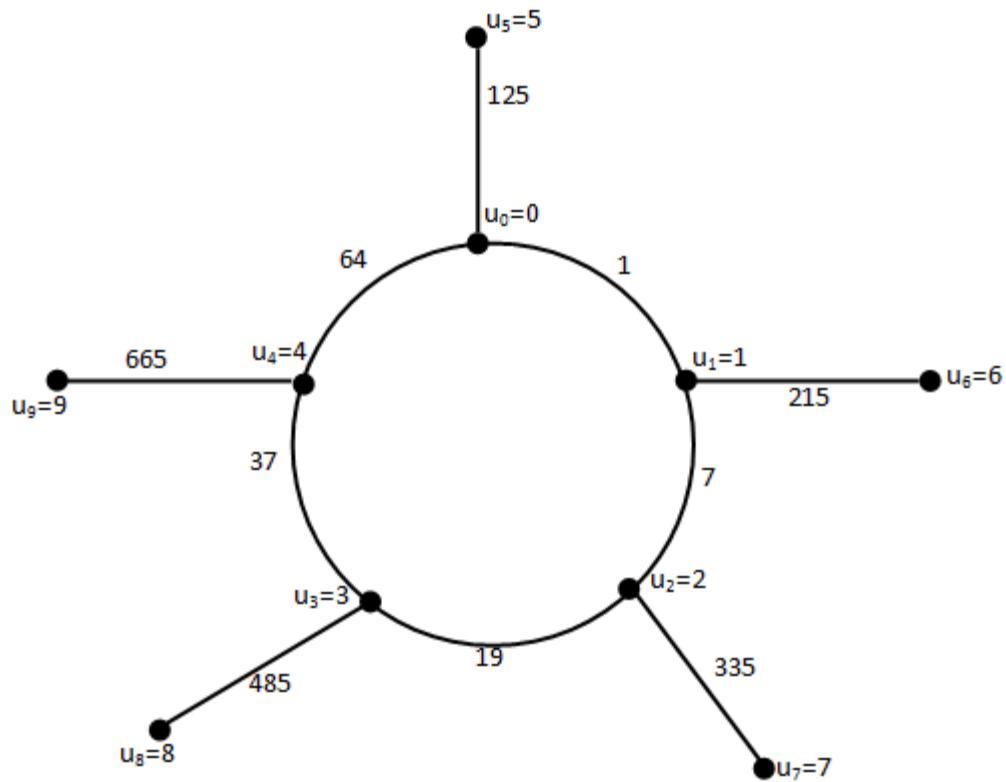
$$\begin{aligned}
 \text{(i)} \quad f^*(u_iu_{i+1}) &= \bigcup_{i=0}^{n-1} |(f(u_i))^3 - (f(u_{i+1}))^3| \\
 &= \bigcup_{i=0}^{n-1} (3i^2 + 3i + 1) \\
 &= \{1, 7, 19, 37\}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad f^*(u_{n-1}u_0) &= (n-1)^3 \\
 &= 64.
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad f^*(u_iu_{n+i}) &= \bigcup_{i=0}^{n-1} |(f(u_i))^3 - (f(u_{n+i}))^3| \\
 &= \bigcup_{i=0}^{n-1} (15i^2 + 75i + 125) \\
 &= \{125, 215, 335, 485, 665\}
 \end{aligned}$$

Here all the edges are distinct. Hence the Sunlet graph S_n admits a Cube difference labeling.

Example 2.8: The Sunlet graph S_5 is a Cube difference graph.

**Theorem: 2.9**

A Sparkler graph P_{m+n} admits a Cube difference labeling.

Proof:

Let P_{m+n} be a Sparkler graph. Let $|V(G)|=m+n$ and $|E(G)|=m+n-1$.

The mapping $f: V(G) \rightarrow \{0, 1, 2, \dots, n-1\}$ is defined by $f(u_i) = i$, $1 \leq i \leq m$

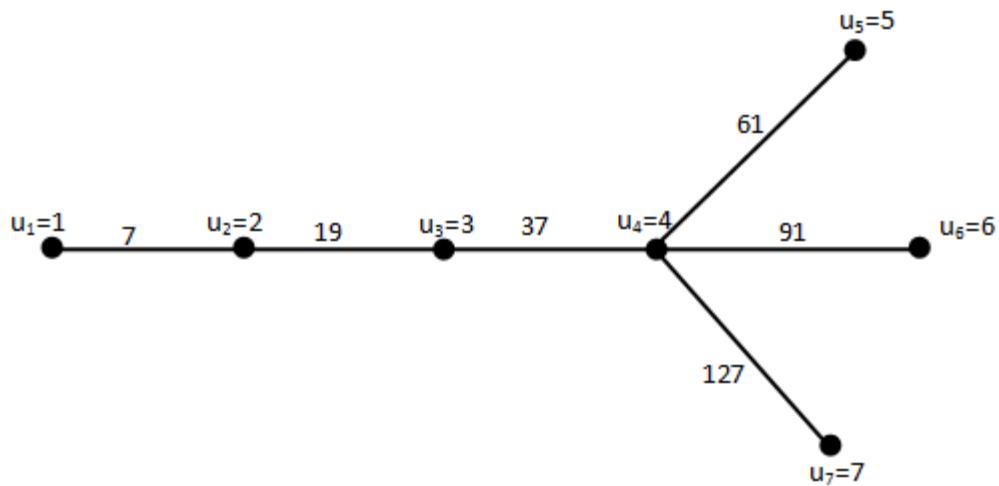
and $f(u_j) = m+1 - j$, $m+1 \leq j \leq 2n+1$, and the induced function, $f^*: E(G) \rightarrow N$ is defined by, and here the sets are, $E_1 = \{u_i u_{i+1} / 1 \leq i \leq m-1\}$, $E_2 = \{u_i v_j / i = m, m+1 \leq j \leq 2n+1\}$ and the edge labeling are

$$(i) \quad f^*(u_i u_{i+1}) = \bigcup_{i=1}^m |(f(u_i))^3 - (f(u_{i+1}))^3| \\ = \bigcup_{i=1}^m (3i^2 + 3i + 1) \\ = \{7, 19, 37\}$$

$$(ii) \quad f^*(u_i u_j) = |(f(u_i))^3 - (f(u_j))^3|, i = m \text{ and } m+1 \leq j \leq n \\ = \bigcup_{i=m+1}^{2n+1} (3i^2 + 3i + 1) \\ = \{61, 91, 127\}$$

Here all the edges are distinct. Hence the Sparkler graph P_{m+n} admits a Cube difference labeling.

Example 2.10: The Sparkler graph P_4+3 is a Cube difference graph.

**Theorem: 2.11**

The Fan graph F_n admits a Cube difference labeling.

Proof:

Let F_n be a Fan graph. Let $|V(G)|=n+1$ and $|E(G)|=2n-1$.

The mapping $f:V(G) \rightarrow \{0,1,2,\dots,n-1\}$ is defined by $f(u)=0$ and $f(u_i)=i$, $1 \leq i \leq n$ and the induced function $f^*:E(G) \rightarrow N$ is defined by, and here the sets are,

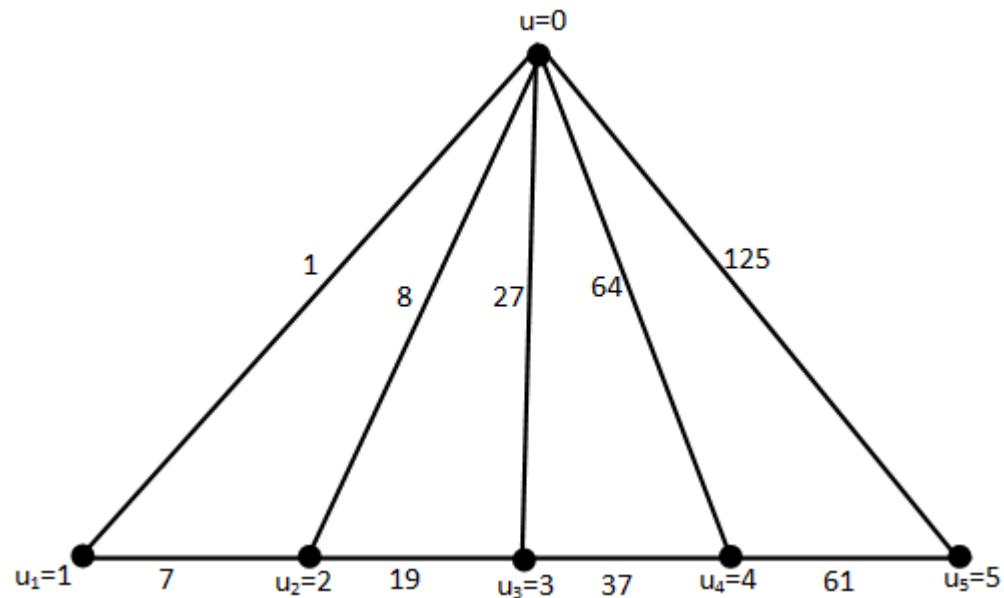
$E_1=\{u_iu_{i+1} / 1 \leq i \leq n-1\}$ and $E_2=\{uu_i / 1 \leq i \leq n\}$ and the edge labelings are,

$$(i) \quad f^*(u_iu_{i+1}) = \bigcup_{i=1}^{n-1} |(f(u_i))^3 - (f(u_{i+1}))^3| \\ = \bigcup_{i=1}^{n-1} (3i^2 + 3i + 1) \\ = \{7, 19, 37, 61\}$$

$$(ii) \quad f^*(uu_i) = \bigcup_{i=1}^n |(f(u))^3 - (f(u_i))^3| \\ = \bigcup_{i=1}^n (i)^3 \\ = \{1, 8, 27, 64, 125\}$$

Here all the edges are distinct. Hence the Fan graph F_n admits a Cube difference labeling.

Example 2.12: The Fan graph F_5 is a Cube difference graph.

**Theorem: 2.13**

A Triangular Snake graph T_n admits a Cube difference labeling.

Proof:

Let T_n be a Triangular Snake graph. Let $|V(G)|=2n+1$ and $|E(G)|=3n$.

The mapping $f:V(G) \rightarrow \{0,1,2,\dots,2n-1\}$ is defined by $f(u_i)=2i$, $0 \leq i \leq n-1$ and $f(v_i)=2i+1$, $0 \leq i \leq n-1$ and the induced function, $f^*:E(G) \rightarrow N$ is defined by, and here the sets are, $E_1=\{v_i v_{i+1} / 0 \leq i \leq n-1\}$, $E_2=\{u_i v_i / 0 \leq i \leq n-1\}$ and $E_3=\{u_i v_{i+1} / 0 \leq i \leq n-1\}$ and the edge labelings are,

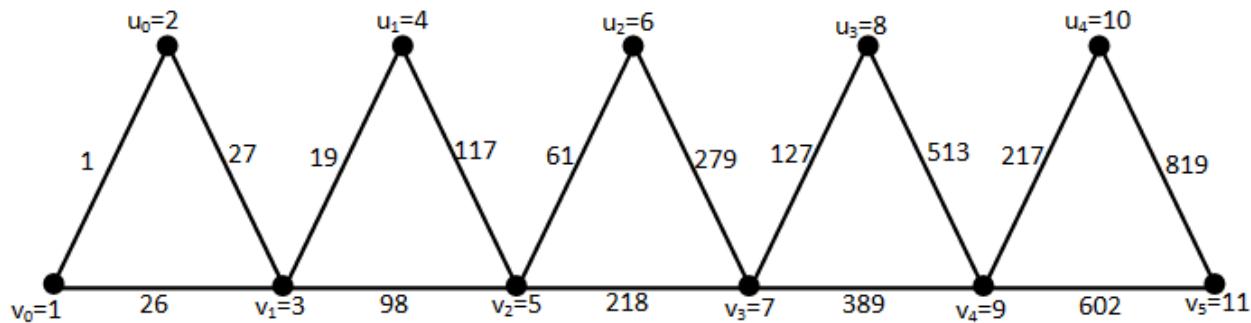
$$\begin{aligned} (i) \quad f^*(v_i v_{i+1}) &= \bigcup_{i=0}^{n-1} |(f(v_i))^3 - (f(v_{i+1}))^3| \\ &= \bigcup_{i=0}^{n-1} |(2(i+1))^3 - (2(i+1)+1)^3| \\ &= \bigcup_{i=0}^{n-1} (24i^2 + 48i + 26) \\ &= \{26, 98, 218, 386, 602\}. \end{aligned}$$

$$\begin{aligned} (ii) \quad f^*(u_i v_i) &= \bigcup_{i=0}^{n-1} |(f(u_i))^3 - (f(v_i))^3| \\ &= \bigcup_{i=0}^{n-1} |(2i)^3 - (2i+1)^3| \\ &= \bigcup_{i=0}^{n-1} (12i^2 + 6i + 1) \\ &= \{1, 19, 61, 127, 217\} \end{aligned}$$

$$\begin{aligned} (iii) \quad f^*(u_i v_{i+1}) &= \bigcup_{i=0}^{n-1} |(f(u_i))^3 - (f(v_{i+1}))^3| \\ &= \bigcup_{i=0}^{n-1} (36i^2 + 54i + 27) \\ &= \{27, 117, 279, 513, 819\} \end{aligned}$$

Here all the edges are distinct. Hence the Triangular Snake graph T_n admits a Cube difference labeling.

Example 2.14: The Triangular Snake graph T_5 is a Cube difference graph.



Theorem: 2.15

The $Z-P_n$ graph admits a Cube difference labeling.

Proof:

Let $Z-P_n$ be a graph. Let $|V(G)|=2n$. The mapping $f:V(G) \rightarrow \{0,1,2,\dots,2n-1\}$ is defined by $f(u_i)=2i$, $0 \leq i \leq n-1$ and $f(v_i)=2i+1$, $0 \leq i \leq n-1$ and the induced function $f^*:E(G) \rightarrow N$ is defined by, and here the sets are,

$E_1=\{u_i u_{i+1} / 0 \leq i \leq n-1\}$, $E_2=\{v_i v_{i+1} / 0 \leq i \leq n-1\}$ and $E_3=\{v_i u_{i+1} / 0 \leq i \leq n-1\}$ and the edges labelings are

$$\begin{aligned} (i) \quad f^*(u_i u_{i+1}) &= \bigcup_{i=0}^{n-1} |(f(u_i))^3 - (f(u_{i+1}))^3| \\ &= \bigcup_{i=0}^{n-1} (24i^2 + 24i + 8) \\ &= \{8, 56, 152, 296\} \end{aligned}$$

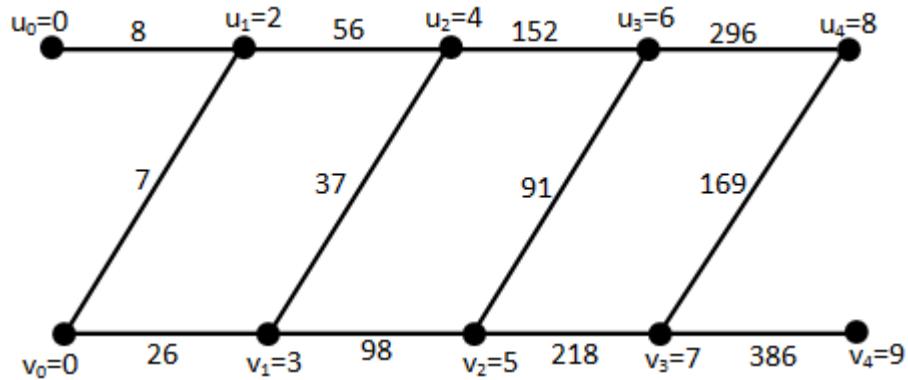
$$\begin{aligned} (ii) \quad f^*(v_i v_{i+1}) &= \bigcup_{i=0}^{n-1} |(f(v_i))^3 - (f(v_{i+1}))^3| \\ &= \bigcup_{i=0}^{n-1} (24i^2 + 48i + 26) \\ &= \{26, 98, 218, 386\} \end{aligned}$$

$$(iii) \quad f^*(v_i u_{i+1}) = \bigcup_{i=0}^{n-1} |(f(v_i))^3 - (f(u_{i+1}))^3|$$

$$\begin{aligned}
 &= \bigcup_{i=0}^{n-1} (12i^2 + 18i + 7) \\
 &= \{7, 37, 91, 169\}
 \end{aligned}$$

Here all the edges are distinct. Hence $Z-P_n$ admits a Cube difference labeling.

Example 2.16: The $Z-P_5$ graph is a Cube difference graph.



III. CONCLUSION

In this paper the Special graphs, are investigated for the Cube difference labeling. This labeling can be verified for some other graphs.

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