Investigating RW-Separation Axioms: An Analytical Approach in Topological Spaces

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ABSTRACT

The aim of this paper is to introduce and study two new classes of spaces, namely Rw-normal and rw- regular spaces and obtained their properties by utilizing rw-closed sets.

KEYWORDS: Rw- closed set, Rw-continuous function,Rw-Separation axioms.

I. INTRODUCTION

Maheshwari and Prasad[8] introduced the new class of spaces called s-normal spaces using semi-open sets. It was further studied by Noiri and Popa[10],Dorsett[6] and Arya[1]. Munshi[9], introduced g-regular and g- normal spaces using g-closed sets of Levine[7]. Later, Benchalli et al [3] and Shik John[12] studied the concept of g* - preregular, g* - pre normal and w- normal, w-regular spaces in topological spaces. Recently, Benchalli et al [2,11] introduced and studied the properties of rw-closed sets and rw-continuous functions.

II. PRELIMINARIES

Throughout this paper space (X, τ) and (Y, σ) (or simply X and Y) always denote topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, Cl(A), Int(A), A^c , and α-Cl(A), denote the Closure of A, Interior of A and Compliment of A and α-closure of A in X respectively.

Definition 2.1: A subset A of a topological space (X, τ) is called

i. W-closed set[12] if cl(A) ⊆ U whenever A⊆ U and U is semi-open in X.
ii. Generalized closed set(briefly g-closed) [7] if cl(A)⊆U whenever A ⊆ U and U is open in X.

Definition 2.2 : A topological space X is said to be a

1. g-regular[10], if for each g-closed set F of X and each point x ∉F,there exists disjoint open sets U and V such that F⊆U and x ϵ V .
2. α - regular [4], if for each α - closed set F of X and each point x ∉ F, there exists disjoint α - open sets U and V such that F ⊆ V and x ϵ U.
3. w-regular[12], if for each closed set F of X and each point x ∉ F, there exists disjoint w-open sets U and V such that F⊆U and xϵ V.

Definition 2.3. A topological space X is said to be a

1. g- normal [10], if for any pair of disjoint g-closed sets A and B, there exists disjoint open sets U and V such that A⊆U and B⊆V .
2. α-normal [4], if for any pair of disjoint $\alpha -$ closed sets A and B, there exists dis-joint α -open sets U and V such that A⊆U and B⊆V .
3. w-normal [12], if for any pair of disjoint w -closed sets A and B, there exists disjoint open sets U and V such that A ⊆ U and B⊆V.

Definition 2.4: [2] A topological space X is called T_{rw} - space if every rw-closed set in it is closed set.

Definition 2.5:A map f: (X, τ) ⟶ (Y, τ) is said to be

i. rw-continuous map[11]if f^{-1}(V)is a rw-closed set of (X, τ) for every closed set V of (Y, τ).
ii. rw-irresolute map[11]if f^{-1}(V)is a rw-closed set of (X, τ) for every rw-closed set V of (Y, τ).

III. RW -REGULAR SPACES

In this section, we introduce a new class of spaces called rw-regular spaces using Rw-closed sets and obtain some of their characterizations.

Definition 3.1. A topological space X is said to be rw-regular if for each rw closed set F and a point x ∉ F, there exist disjoint open sets G and H such that F⊆G and x ϵ H.
We have the following interrelationship between rw-regularity and regularity.

Theorem 3.2. Every rw-regular space is regular.

Proof: Let X be a rw-regular space. Let F be any closed set in X and a point x∉X such that x∉F. By [2], F is rw-closed and x ∉ F. Since X is a rw-regular space, there exists a pair of disjoint open sets G and H such that F ⊆ G and x ϵ H. Hence X is a regular space.

Remark 3.3. If X is a regular space and T_{rw} space, then X is rw regular We have the following characterization.

Theorem 3.4. The following statements are equivalent for a topological space X
(i) X is a rw regular space
(ii) For each x ϵ X and each rw-open neighbourhood U of x there exists an open neighbourhood N of x such that cl(N)⊆U.

Proof: (i) → (ii): Suppose X is a rw regular space. Let U be any rw neighbour-hood of x. Then there exists rw open set G such that x ϵ G ⊆U. Now X –G is rw closed set and x ∉ X - G. Since X is rw regular, there exist open sets M and N such that X -G⊆M, x ϵ N and M ∩ N = φ and so N ⊆X-M. Now cl(N) ⊆ cl(X -M) = X -M and X -M ⊆ M. This implies X -M⊆ U. Therefore cl(N)⊆U.

(ii) →(i): Let F be any rw closed set in X and x ϵ X -F and X - F is a Rw-open and so X - F is a rw-neighbourhood of x. By hypothesis, there exists an open neighbourhood N of x such that x ϵ N and cl(N) ⊆ X - F. This implies F ⊆ X - cl(N) is an open set containing F and N ∩ f(X - cl(N)= φ . Hence X is rw- regular space.
We have another characterization of rw-regularity in the following.

Theorem 3.5: A topological space X is rw-regular if and only if for each rw-closed set F of X and each x ϵ X - F there exist open sets G and H of X such that x ϵ G, F⊆H and cl(G) ∩ cl(H) = ∅.

Proof: Suppose X is rw-regular space. Let F be a rw-closed set in X with x ∉ F.Then there exists open sets M and H of X such that x ϵ M, F ⊆H and M∩H =∅. This implies M∩cl(H) = ∅.As X is rw-regular, there exist open sets U and V such that x ϵ U, cl(H)⊆V and U∩V = ∅. so cl(U)∩V = ∅.Let G = M ∩ U, then G and H are open sets of X such that xϵG, F ⊆ H and cl(H) ∩ cl(H) =∅ .
Conversely, if for each rw-closed set F of X and each x ϵ X -F there exists open sets G and H such that x ϵ G, F⊆ H and cl(H) ∩cl(H) = ∅ .This implies x ϵ G,F⊆H and G ∩ H = ∅. Hence X is rw- regular.
Now we prove that rw- regularity is a heriditary property.

Theorem 3.6. Every subspace of a rw-regular space is rw-regular.

Proof: Let X be a rw- regular space. Let Y be a subspace of X. Let x ϵ Y and F be a rw-closed set in Y such that x∉F. Then there is a closed set and so rw-closed set A of X with F = Y ∩ A and x ∉A. Therefore we have x ϵ X, A is rw – closed in X such that x∉A. Since X is rw- regular, there exist open sets G and H such that x ϵ G, A⊆H and G∩H = φ. Note that Y ∩ G and Y ∩ H are open sets in Y .Also x ϵ G and x ϵ Y, which implies x ϵ Y ∩G and A ⊆ H implies Y∩ G ⊆Y ∩ H,F⊆Y ∩ H. Also (Y ∩ G) ∩ (Y ∩ H) = φ. Hence Y is rw-regular space.
We have yet another characterization of rw -regularity in the following.

Theorem 3.7 : The following statements about a topological space X are equivalent:
(i) X is rw -regular
(ii) For each x ϵ X and each rw-open set U in X such that x ϵ U there exists an open set V in X such that x ϵ V⊆cl(V)⊆U.
(iii) For each point x ϵX and for each rw-closed set A with x ∉ A, there exists an open set V containing x such that cl(V)∩A = φ.

Proof: (i) $\rightarrow$(ii): Follows from Theorem 3.5.
(ii) $\rightarrow$(iii): Suppose (ii) holds. Let $x \in X$ and A be an rw-closed set of X such that $x \notin A$.Then X - A is a rw-open set with $x \in X - A$. By hypothesis, there exists an open set V such that $x \in V \subseteq cl(V) \subseteq X - A$. That is $x \in V$, $V \subseteq cl(A)$ and $cl(A) \subseteq X - A$. So $x \in V$ and $cl(V) \cap A = \varphi$.
(iii) $\rightarrow$ (i): Let $x \in X$ and U be an rw-open set in X such that $x \in U$. Then X - U is an rw closed set and $x \notin X - U$. Then by hypothesis, there exists an open set V containing x such that cl(A) ∩(X -U) = Á. Therefore $x \in V$, $cl(V) \subseteq U$ so $x \in V \subseteq cl(V) \subseteq U$.
The invariance of rw-regularity is given in the following.

Theorem 3.8: Let $f : X \rightarrow Y$ be a bijective, rw-irresolute and open map from a rw- regular space X into a topological space Y , then Y is rw-regular.

Proof: Let $y \in Y$ and F be a rwclosed set in Y with $y \notin F$. Since F is rw- irresolute, $f^{-1}(F)$ is rw-closed set in X. Let f(x) = y so that $x = f^{-1}(y)$ and $x \notin f^{-1}(F)$. Again X is rw-regular space, there exist open sets U and V such that $x \in U$ and $f^{-1}(F) \subseteq G$, $U \cap V = \varphi$. Since f is open and bijective, we have y f(U),$F \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) = f(\varphi) = \varphi$. Hence Y is rw-regular space.

Theorem 3.9. Let $f : X \rightarrow Y$ be a bijective, rw-closed and open map from a topological space X into a rw-regular space Y . If X is T_{rw} space, then X is rw-regular.

Proof: Let $x \in X$ and F be an rw-closed set in X with $x \notin F$. Since X is T_{rw} space,F is closed in X. Then f(F) is rwclosed set with $f(x) \notin f(F)$ in Y , since f is rw- closed. As Y is rw-regular, there exist open sets U and V such that $x \in U$ and $f(x) \in U$ and $f(F) \subseteq V$. Therefore $x \in f^{-1}(U)$ and $F \subseteq f^{-1}(V)$. Hence X is rw-regular space.

Theorem 3.10. If $f : X \rightarrow Y$ is w-irresolute, continuous injection and Y is rw-regular space,then X is rw- regular.

Proof: Let F be any closed set in X with $x \notin F$. Since f is w-irresolute, f is rw- closed set in Y and $f(x) \in f(F)$. Since Y is rw- regular,there exists open sets U and V such that $f(x) \in U$ and $f(F) \subseteq V$. Thus $x \in f^{-1}(U)$,$F \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \varphi$. Hence X is rw- regular space.

IV. RW-NORMAL SPACES

In this section, we introduce the concept of rwnormal spaces and study some of their characterizations.

Definition 4.1. A topological space X is said to be rw-normal if for each pair of disjoint rw- closed sets A and B in X, there exists a pair of disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$
We have the following interrelationship.

Theorem 4.2. Every rw-normal space is normal.

Proof: Let X be a rw-normal space. Let A and B be a pair of disjoint closed sets in X. From [2], A and B are rw-closed sets in X. Since X is rw-normal, there exists a pair of disjoint open sets G and H in X such that $A \subseteq G$ and $B \subseteq H$. Hence X is normal.

Remark 4.3. The converse need not be true in general as seen from the following
example.

Example 4.4. Let X = Y ={a,b,c,d},τ ={X, ∅,{a},{c},{a,c},{b,c,d}} Then
the space X is normal but not rw - normal, since the pair of disjoint rw - closed sets namely, A = {a,d} and B = {b,c} for which there do not exists disjoint open sets G and H such that $A \subseteq G$ and $B \subseteq H$.

Remark 4.5.:If X is normal and T_{rw}-space, then X is rw-normal.
Hereditary property of rw- normality is given in the following.

Theorem 4.6. A rw - closed subspace of a rw - normal space is rw -normal. We have the following characterization.

Theorem 4.7. The following statements for a topological space X are equivalent:
(i) X is rw- normal
(ii) For each rw - closed set A and each rw - open set U such that
$A \subseteq U$, there exists an open set V such that $A \subseteq V \subseteq cl(V) \subseteq U$

(iii) For any rw-closed sets A, B, there exists an open set V such that A⊆V and cl(V)∩B = φ.
(iv) For each pair A, B of disjoint rw-closed sets there exist open sets U and V such that A ⊆ U,B ⊆V and cl(U) ∩ cl(V) = φ.

Proof: (i) →(ii): Let A be a rw-closed set and U be a rw-open set such that A⊆ U.Then A and X - U are disjoint rw-closed sets in X. Since X is rw-normal , there exists a pair of disjoint open sets V and W in X such that A ⊆ V and X -U ⊆W. Now X -W ⊆ X - (X -U), so X -W⊆ U also V∩ W = φ. implies V ⊆ X -W, so cl(V) ⊆ cl(X -W) which implies cl(V) ⊆ X -W. Therefore cl(V) ⊆ X -W⊆ U. So cl(V) ⊆ U. Hence A ⊆ V ⊆ cl(V) ⊆ U.

(ii) → (iii): Let A and B be a pair of disjoint rwclosed sets in X. Now A∩ B = φ,so A⊆ X -B, where A is rw-closed an⊆ d X - B is rw-open . Then by (ii) there exists an open set V such that A⊆V⊆cl(V)⊆ X - B. Now cl(V) ⊆ X - B implies cl(V) ∩ B = φ . Thus A ⊆ V and cl(V)∩B = φ.

(iii) → (iv): Let A and B be a pair of disjoint rw-closed sets in X.Then from (iii) there exists an open set U such that A⊆U and cl(U) ∩ B = φ. Since cl(V) is closed, so rw-closed set.Therefore cl(V) and B are disjoint rwclosed sets in X. By hypothesis, there exists an open set V , such that B⊆V and cl(U) ∩cl(V) = φ.

(iv) →(i): Let A and B be a pair of disjoint rw-closed sets in X.Then from (iv) there exist an open sets U and V in X such that A⊆U, B⊆V and cl(U) ∩ cl(V) = φ. So A ⊆ U , B⊆V and U∩V = φ.Hence X rw-normal.

Theorem 4.8. Let X be a topological space. Then X is rw-normal if and only if for any pair A, B of disjoint rw-closed sets there exist open sets U and V of X such that A⊆U,B⊆V and cl(U) ∩cl(V) = φ.

Theorem 4.9. Let X be a topological space. Then the following are equivalent:
(i) X is normal
(ii) For any disjoint closed sets A and B, there exist disjoint rw - open sets U and V
such that A⊆U,B⊆V .
(iii) For any closed set A and any open set V such that A⊆ V, there exists an rw -open set U of X such that A⊆U⊆ αcl(U) ⊆ V .

Proof:
(i) → (ii): Suppose X is normal. Since every open set is rw-open [2], (ii) follows.
(ii) → (iii): Suppose (ii) holds. Let A be a closed set and V be an open set containing A. Then A and X - V are disjoint closed sets. By (ii), there exist disjoint rw- open sets U and W such that A⊆U and X - V ⊆ W, since X - V is closed, so rw - closed. From [2], we have X -V⊆ α-int(W) and U ∩ α-int(W) = φ.and so wehave α-cl(U) ∩ α-int(W) = φ. Hence A ⊆ U ⊆ α-cl(U) ⊆ X – α-int(W)⊆V . Thus A ⊆ U ⊆ α-cl(U) ⊆ V .
(iii) → (i): Let A and B be a pair of disjoint closed sets of X.Then A ⊆ X - B and X -B is open. There exists a rw - open set G of X such that A ⊆ G ⊆ α-cl(G) ⊆ X-B. Since A is closed, it is w- closed, we have A ⊆ α -int(G). Take U = int(cl(int(α-int(G))))and V = int(cl(int(X –α-cl(G)))). Then U and V are disjoint open sets of X such that A ⊆ U and B ⊆ V . Hence X is normal.

We have the following characterization of rw - normality and rw- normality.

Theorem 4.10: Let X be a topological space. Then the following are equivalent:
(i) X is α-normal.
(ii) For any disjoint closed sets A and B, there exist disjoint rw - open sets U and V such that A⊆ U,B⊆V and U∩ V = φ.

Proof:
(i) →(ii): Suppose X is α- normal. Let A and B be a pair of disjoint closed sets of X. Since X is α -normal,there exist disjoint $\alpha -$ open sets U and V such that A⊆U and B⊆V and U ∩ V = φ.
(ii) → (i):Let A and B be a pair of disjoint closed sets of X.The by hypothesis there exist disjoint rw- open sets U and V such that A⊆U and B ⊆ V and U ∩V = φ .Since from [2], A⊆ α-intU and B ⊆ $\alpha -$ int(V)and α –int U∩ α -intV = φ. Hence X is α -normal.

Theorem 4.11. Let X bea α- normal, then the following hold good:
(i)For each closed set A and every rw - open set B such that A⊆B ther exists a α open set U such that A⊆U⊆ α-cl(U)⊆ B.

(ii) For every rw-closed set A and every open set B containing A, there exist a α-open set U such that A⊆U⊆ α-cl(U)⊆B

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