Applying Differential Transform Method (DTM) to Analyze the Cholera Carrier Epidemic Model

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ABSTRACT

In this paper, Differential Transform Method (DTM) is applied to the deterministic mathematical model of cholera carrier epidemic. The model is transformed using DTM operational properties, hence, the power series of the model system is generated and also an approximate solution of the model was established. The accuracy of DTM is demonstrated against Fehlberg Runge-Kutta method with order four interpolant (RKF-45) numerical solution and it demonstrated high accuracy of the results. Plotted DTM solution is found to be in good agreement with the popular Runge-Kutta solution.

KEYWORDS: Cholera carrier, Epidemic model, Runge-Kutta, Differential transform.

I. INTRODUCTION

Differential transform method was first introduced in 1986 by Zhou to study the initial value problems in electrical circuit for a computation in linear and non-linear systems. Since that time, the technique has been used to computes equation such as Differential algebraic equations, Fractional differential equation, Lane-Emden type equations and Schrodinger equations [2]. Currently, the application of DTM is practice in the mathematical Biology and mathematical epidemiology research.

The recent applications of DTM in some literature [19] presented the differential transform method on computation of the model for biological species living together, they applied the DTM to finds the numerical solution of the problem. In order to produce simulations [10] proposed new modification of the DTM to study a SIRC influenza model, they apply different initial conditions and parameter values for the basic reproduction number; [2] study the application of differential transform method and variational iteration method in the numerical solution of SIR model, their result finding shows that, both methods are accurate and efficient for computation of ODEs. A new method for computation of epidemics model was discussed by [4] their results are compared with the results obtained from different numerical method. [5] study the application of the DTM to used 4-5th order Runge-Kutta method with degree four interpolant (RKF45) and they obtained the high accuracy results in the numerical solution of the lake system problem. [18] apply DTM to demonstrated solution of some non-linear differential equations [8], [12], [1], [4] and [19].

In this work we aim to present the application of differential transform method to the proposed deterministic mathematical model of cholera carrier epidemic in an effort to provide solution in curtailing the spread of the disease.

Concept of the Differential Transforms Method

Given the function $b(t)$ in Taylor series about a point $t=0$, such that $b(t)=\sum_{i=0}^{k}\frac{t^k}{k!}\left[\frac{d^k b}{dt^k}\right]_{t=0}$, then the differential transform of $b(t)$ is given as, $B(k)=\frac{1}{k!}\left[\frac{d^k b}{dt^k}\right]_{t=0}$ and the inverse differential transform is $b(k)=\sum_{k=0}^{\infty}t^k B(k)$.

Some operational properties of Differential Transform Method

Given two uncorrelated functions $u(t)$ and $v(t)$ with time t, then $U(k)$ and $V(k)$ are the transformed functions corresponding to $u(t)$ and $v(t)$, respectively. Then, the following properties hold:

1 If $b(t)=u(t)\pm v(t), \quad Then \quad B(k)=U(k)\pm V(k)$

2 If $b(t)=\alpha u(t), \quad Then \quad B(k)=\alpha U(k)$

3 If $b(t)=\dfrac{du(t)}{dt}, \quad Then \quad B(k)=(k+1)U(k+1)$

4 If $b(t)=\dfrac{d^2u(t)}{dt^2}, \quad Then \quad B(k)=(k+1)(k+2)U(k+2)$

5 If $b(t)=\dfrac{d^mu(t)}{dt^m}, \quad Then \quad B(k)=(k+1)(k+2)...(k+m)U(k+m)$

6 If $b(t)=u(t)v(t), \quad Then \quad B(k)=\sum_{l=0}^{k}V(l)U(k-l)$

7 If $b(t)=1, \quad Then \quad B(k)=\delta(k)$

8 If $b(t)=t, \quad Then \quad B(k)=\delta(k-1)$

9 If $b(t)=t^m, \quad Then \quad B(k)=\delta(k-m), \quad \delta(k-m)=\begin{cases}1 & if \quad k=0\\ 0 & if \quad k\neq m\end{cases}$

10 If $b(t)=\exp(\lambda t), \quad Then \quad B(k)=\dfrac{\lambda^k}{k!}$

11 If $b(t)=(1+t)^m, \quad Then \quad B(k)=\dfrac{m(m-1)...(m-k+1)}{k!}$

II. MODEL FORMULATIONS

The following assumptions are considered to design the cholera [3], [6], [15] carrier [7] model.

i. Allowing recruitment of immigrant with cholera carrier.

ii. Using combine incidence rates of the form $\left(\beta_1 C+\beta_2 I+\dfrac{\beta_3 P}{1+\alpha P}\right)$.

iii. Transmission occurred through direct contact with infectious or contact with carrier individuals with V.cholerae.

iv. Assume that displaced individuals do not form separate settlements.

Using mentioned assumptions, the schematic diagram below establish the interaction between different populations:

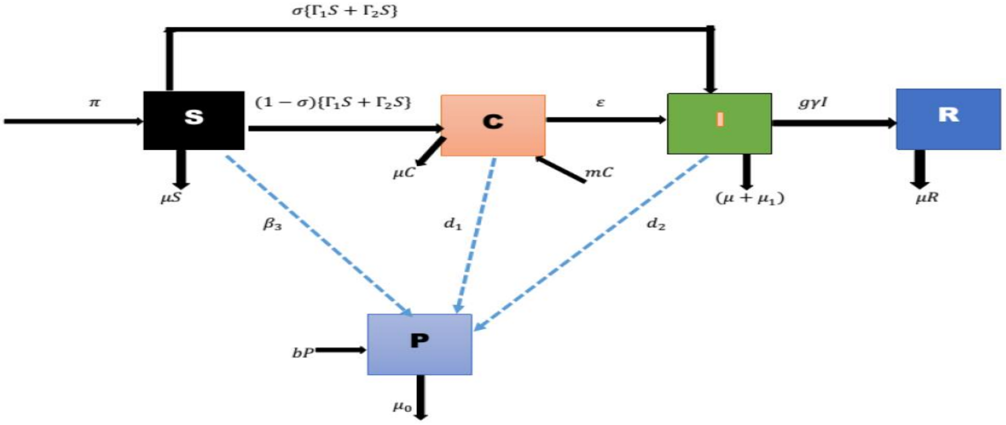


Figure 1: Flow diagram of the cholera carrier model

Thus, the model formulation is governed by the following system of non -linear differential equations:

$$\left.\begin{aligned}
\frac{dS}{dt} &= \pi - \left[\beta_1 C + \beta_2 I + \frac{\beta_3 P}{1+\alpha P}\right] S(t) - \mu S(t) \\
\frac{dC}{dt} &= (1-\sigma)\left[\beta_1 C + \beta_2 I + \frac{\beta_3 P}{1+\alpha P}\right] S - (\mu + \grave{o} - m) C(t) \\
\frac{dI}{dt} &= \grave{o} C + \sigma \left[\beta_1 C + \beta_2 I + \frac{\beta_3 P}{1+\alpha P}\right] S - (\mu + \mu_1 + g\gamma) I(t) \\
\frac{dR}{dt} &= g\gamma I(t) - \mu R(t) \\
\frac{dP}{dt} &= d_1 C + d_2 I - (\mu_0 - b) P
\end{aligned}\right\} \quad (1)$$

Subject to the initial conditions

$S(0) = S_0, C(0) = C_0, I(0) = I_0, R(0) = R_0. P(0) = P_0$

Table 1: Description of Variables, Parameters, and Hypothetical values with their sources

Variables	**Description of variables**
$S(t)$	Susceptible Population
$C(t)$	Carrier Population
$I(t)$	Symptomatic Population
$R(t)$	Removed Population
$P(t)$	Pathogen Population

Parameters	**Description of Parameters**	**Value**	**Sources**
π	Birth rate	0.6	Estimate
σ	Cholera carrier symptomatic	0-1	Varied
μ	Natural death	0.2	Estimate
β_1	Force of infection in susceptible	$1.5\beta_2$	[11]
β_2	Force of infection in carrier	0.05	[14]
β_3	Force of infection in pathogen	0.05	[14]
m	Immigrants with carrier	0-1	Varied
μ_1	Death due to V. Cholera	0.015	[20]
μ_0	Death of Vibrios	0.02	Estimate
b	Growth of V.cholera	0.01	[20]
$g\gamma$	Clearance rate due to treatment	0.2	[9]
d_1	V. cholera Contribution	0-1	Varied
d_2	V. cholera Contribution	0.01	[16]
α	Saturation constant	0.02	[14]

III. SOLUTION OF CHOLERA CARRIER MODEL USING DIFFERENTIAL TRANSFORM METHOD

Using operational properties (1), (2), (3), (6) and (7) of DTM in subsection (1.2) and applying it into the model system (1), the first equation is transformed into

$$(k+1)S(k+1) = \pi\cdot\delta(k,0) - \left[\beta_1 \sum_{l=0}^{k}[S(l)C(k-l)] + \beta_2 \sum_{l=0}^{k}[S(l)M(k-l)] + \beta_3 \sum_{l=0}^{k}[S(l)P(k-l)]\right] - \mu S(k)$$

Dividing through by $(k+1)$ so as to make $S(k+1)$ the subject of the expression, then

$$S(k+1) = \frac{1}{k+1}\left\{\pi\cdot\delta(k,0) - \left[\beta_1 \sum_{l=0}^{k} S(l)C(k-l) + \beta_2 \sum_{l=0}^{k} S(l)M(k-l) + \beta_3 \sum_{l=0}^{k} S(l)P(k-l)\right] - \mu S(k)\right\}$$

Second equation of model system (1) is transformed into

$$(k+1)C(k+1)=\left[\beta_1\sum_{l=0}^{k}[S(l)C(k-l)]+\beta_2\sum_{l=0}^{k}[S(l)M(k-l)]+\beta_3\sum_{l=0}^{k}[S(l)P(k-l)]\right]-Q_1C(k)$$

This implies

$$C(k+1)=\frac{1}{k+1}\left\{\beta_1\sum_{l=0}^{k}[S(l)C(k-l)]+\beta_2\sum_{l=0}^{k}[S(l)M(k-l)]+\beta_3\sum_{l=0}^{k}[S(l)P(k-l)]-Q_1C(k)\right\}$$

Similarly, $I(t)$

$$(k+1)I(k+1)=\grave{o}C(k)-Q_2I(k)$$

$$\Rightarrow I(k+1)=\frac{1}{k+1}\left\{\grave{o}C(k)-Q_2I(k)\right\}$$

The fourth and the fifth equation of the model system (1) yields

$$R(k+1)=\frac{1}{k+1}\left[g\gamma I(k)-\mu R(k)\right]$$

$$P(k+1)=\frac{1}{k+1}\left[d_1C(k)+d_2I(k)-Q_3P(k)\right]$$

Thus, the system of equations governing the cholera carrier model in (1) is transformed into DTM of the form:

$$S(k+1)=\frac{1}{k+1}\left[\pi\cdot\delta(k,0)-\left[\beta_1\sum_{l=0}^{k}[S(l)C(k-l)]+\beta_2\sum_{l=0}^{k}[S(l)M(k-l)]+\beta_3\sum_{l=0}^{k}[S(l)P(k-l)]\right]-\mu S(k)\right]$$

$$C(k+1)=\frac{1}{k+1}\left[\left[\beta_1\sum_{l=0}^{k}[S(l)C(k-l)]+\beta_2\sum_{l=0}^{k}[S(l)M(k-l)]+\beta_3\sum_{l=0}^{k}[S(l)P(k-l)]\right]-Q_1C(k)\right]$$

$$I(k+1)=\frac{1}{k+1}\left[\grave{o}C(k)-Q_2I(k)\right]$$

$$R(k+1)=\frac{1}{k+1}\left[g\gamma I(k)-\mu R(k)\right]$$

$$P(k+1)=\frac{1}{k+1}\left[d_1C(k)+d_2I(k)-Q_3P(k)\right]$$

Subject to the initial conditions

$$\mathrm{S}(0)=1.89,\ C(0)=0.75,\ I(0)=0.26,\ R(0)=0.1,\ P(0)=20$$

Then, the series solution for each compartment of the cholera carrier model using DTM is obtained by using the initial conditions and the parameter values given by Table 1.

We perform the iteration of the resulting DTM of cholera carrier model and obtained

$$S(1)=-1.7988825,\ C(1)=1.3458825,\ I(1)=0.49210,\ R(1)=0.032,\ P(1)=0.1224$$

$$S(2)=1.028758060,\ C(2)=-1.454516934,\ I(2)=0.4362422500,\ R(2)=0.04601000000,\ P(2)=0.07036662500$$

$$S(3)=-0.3108714174,\ C(3)=0.6786426270,\ I(3)=-0.4482180270,\ R(3)=0.02601548333,\ P(3)=-0.4726431240$$

$$S(4)=0.01791674872,\ C(4)=-0.1550677689,\ I(4)=0.1822311457,\ R(4)=-0.2371167552,\ P(4)=0.01596368139$$

Thus, the series solution of the transformed expressions for S(t), C(t), I(t), R(t) and P(t) are derived as:

$$S(t)=1.89-1.7988825t+1.028758060t^2-0.3108714174t^3+0.01791674872t^4+........$$

$$C(t)=0.75+1.3458825t-1.454516934t^2+0.6786426270t^3-0.1550677689t^4+........$$

$$I(t)=0.26+0.49210t+0.43624225t^2-0.4482180270t^3+0.1822311457t^4+.......$$

$$R(t)=0.1+0.032t+0.04601000000t^2+0.02601548333t^3-0.02371167552t^4+......$$

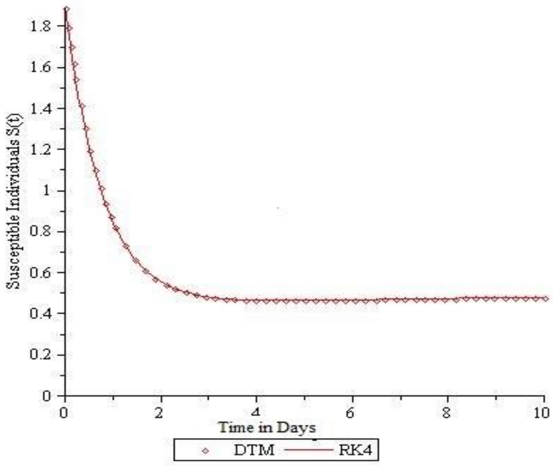
$$P(t)=20-0.1224t+0.07036662500t^2-0.4726431240t^3+0.01596368139t^4+...$$

IV. NUMERICAL SIMULATIONS AND DISCUSSION

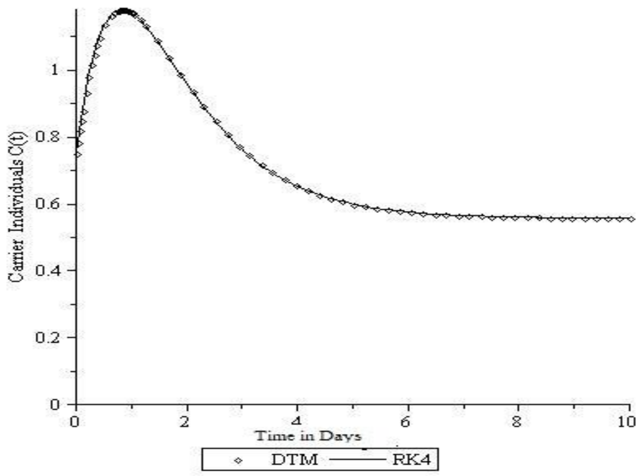
In this section, We present a numerical simulation of model system (1) using a set of parameter values given in Table (1) where S(k), C(k), I(k), R(k) and P(k) are the differential transforms of the corresponding functions S(t), C(t), I(t), R(t) and P(t) respectively with initial conditions given as

S(0) =1.89, C(0) =0.75, I(0) =0.26, R(0) =0.1 and P(0) =20

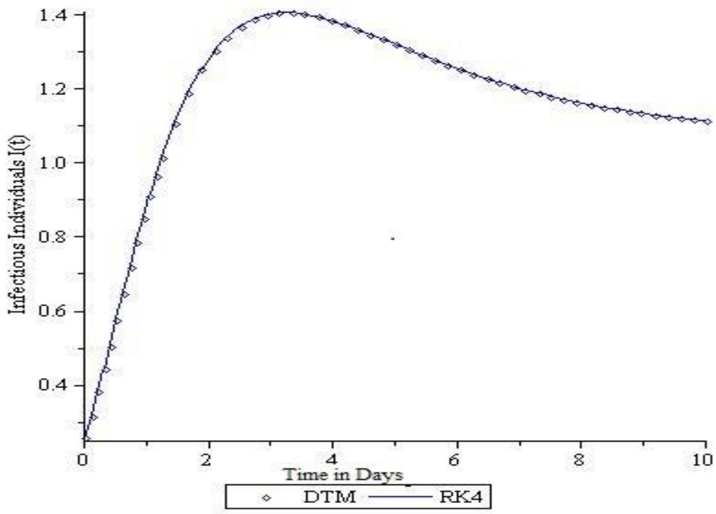
The DTM is demonstrated against maple built-in fourth order Runge-Kutta procedure for the solutions of cholera carrier model. Both methods converges and tends to cholera free equilibrium point so that the disease dies out with time.



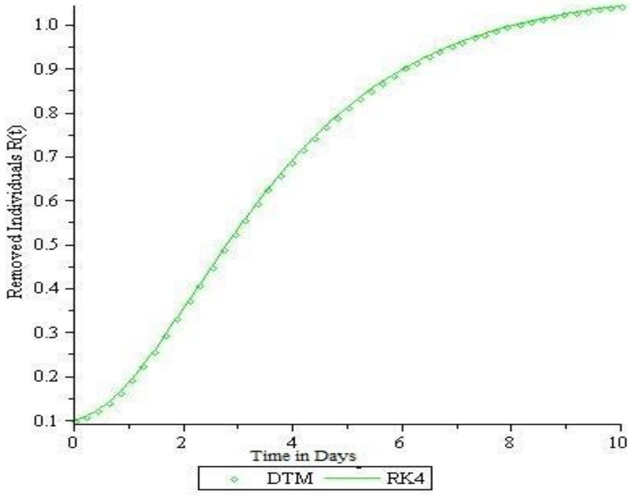
(a)



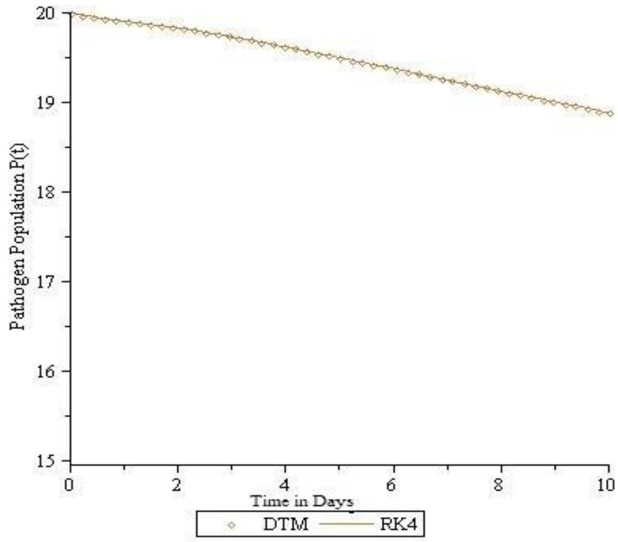
(b)



(c)



(d)



(e)

Figure 2: Simulations (a-e) depicts the comparison between DTM and RK-4 solutions of the model system (1) populations against Time (in days)

V. CONCLUSION

In this paper, solution of cholera carrier epidemic model using differential transform method has been successfully presented and analyzed rigorously. The study achieved the following:

i. Using DTM operational properties, cholera carrier epidemic model is transformed and the power series solutions are generated.
ii. The accuracy of DTM is demonstrated against the maple built-in fourth order Runge-Kutta method.
iii. Plotted DTM solutions are found to be in good agreement with the popular Runge-Kutta solution.
iv. The results revealed that the method has high accuracy and required less computational work compared to the other methods.

In view to that it is recommended that mathematical modeling problems can be solved using DTM because of it is efficiency, reliability, high accuracy, and fast convergence rate. Furthermore, it can be applied directly to problem in linear or non-linear ordinary differential equations

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