An Analytical Study of Quantum Mechanics Through Relativistic Formalisms

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Historically, Paul Dirac developed a relativistic quantum mechanical equation for the electron, which included spin and predicted antiparticles. Nonrelativistic quantum mechanics already existed at the time, as did experimental evidence for spin. It appears that Dirac did not assume the existence of negative energies a priori. In 1930, Erwin Schrodinger, by examining the Dirac equation, suggested the existence of zitterbewegung motion. In 1984, Barut and Zanghi [2], [3] developed a relativistic classical equation for the electron which predicted helical motion, which in turn could explain wavelength and spin. (They started with a classical Lagrangian model in which there was a classical analogue of Zitterbewegung. [3] Their model was later quantized to give an equation similar to that of Dirac´s.)

The purpose of this brief note is to consider whether the ideas of quantum mechanics, in particular a hint of particle wave duality (or at least of an internal frequency), zitterbewegung, non-commuting operators of observables, time evolution by the energy matrix exp(-iHt) etc already appear in Einstein´s 1905 special relativity equation for momentum and energy. No use of quantum mechanical formalism is made, only linear algebra, but it is assumed a priori that both +E and -E are allowed.

Let us first consider the equation: $E^2 = p^2 + m^2$ (1), where m is the rest mass and c is set to 1 throughout . Mathematically, this equation allows for +E and -E, so as a simple model, consider one dimensional motion with both positive and negative energies included. This leads to a 2x2 diagonal matrix with E and -E as eigenvalues. The reason we have moved from the one dimensional scalar equation (1), to a 2x2 space is thus clear at the beginning. Any results, of such a model, therefore, should depend on the existence of both positive and negative energy eigenfunctions. One can see that a number of orthogonal matrices, linear in p and m, when squared give a diagonal matrix with E*E as both eigenvalues. These matrices have +E and -E as eigenvectors. Four such matrices are (with c set to 1)

$$\begin{vmatrix} p & m \\ m & -p \end{vmatrix} \qquad \begin{vmatrix} -p & m \\ m & p \end{vmatrix} \qquad \begin{vmatrix} m & p \\ p & -m \end{vmatrix} \qquad \begin{vmatrix} -m & p \\ p & m \end{vmatrix}$$

Let us consider the first matrix as an eigenvalue equation as similar results hold for the others. (It can be seen that Pauli matrices are contained in these matrices.)

$$p \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + m \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \text{ vector } = E \text{ vector } \quad (2)$$

We can call this matrix H, the energy matrix.

The +E eigenvector is (m, E-p) and the -E, (m, -E-p). The two are orthogonal. A normalization factor for the +E vector is 1/ sqrt[2E(E-p)].

Consider the scalar equation $pv + \text{sqrt}(1 + v^2)\, m = E$ (3) which exists in special relativity. This equation, when compared with (2) suggests that v is identified with $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$ (4)

and $\text{sqrt}(1 + v^2)$ with $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ (5) .

Note: With scalars sqrt($1+v^2$) and 1/sqrt($1+v^2$) are inverses, but such is not the case with the corresponding matrices. The matrix for 1/sqrt($1+v^2$) is

$$\begin{vmatrix} p/m & 1 \\ 1 & -p/m \end{vmatrix}$$

There is nothing new in representing velocity by a matrix, but here it appears classically. Similar matrix math to that presented above is contained in the work of Dirac, but there it is implicitly assumed that one is working with quantum mechanical formalism. In fact, arguments related to zitterbewegung [1], the velocity operator is derived using Heisenberg´s quantum mechanical formalism dx/dt = [H , x], with p in the Dirac equation being proportional to the operator d/dx. No such quantum mechanical formalism is needed here, terms have just been matched, yet we already have an operator picture of what was a scalar v by using the positive-negative energy matrix picture. In other words, ideas that may seem to be related to the world of quantum mechanics (derived first in the nonrelativistic case), may exist directly in the special relativity equation (1). Taking the absolute value of the expectation value of the v matrix with the normalized +E eigenvalue gives v, so the connection seems justified. The velocity operator does not commute with H, the energy matrix, and so it seems that for a particle in an energy state (energy eigenvector) one would not be able to trace the velocity and hence the position exactly, like in classical physics. Equation (1), however, is a classical equation, we have just put it into matrix form with the assumption of positive and negative E. In the scalar equation (1), v is a number. Now, in a +E, -E model, the expectation value of an operator is v. The +E eigenvector is not an eigenvector of the velocity operator. It seems that zitterbewegung exists here. The particle described has an overall <v>, but there is ¨internal¨ motion. To see this more clearly, one can note that the +E eigenvector is equivalent to (sqrt($1+v^2$), sqrt($1-v^2$)). In addition, the velocity vector has eigenvalues +1 and -1, associated with forward and backward motion, and the two component +E eigenvector contains a positive v eigenvector piece and a negative v eigenvector piece. It appears that there is some kind of periodic motion occurring inside the particle associated with the speed of light c, or with a photon. Let us see if these ideas can be made concrete.

Consider the case of the energy matrix with p=0 namely

$$\begin{vmatrix} 0 & m \\ m & 0 \end{vmatrix} \quad (6)$$

This has the positive energy (m) eigenvector of (1,1) with a normalization of sqrt(2).

This vector is the sum of the two velocity eigenvectors, with equal weight. In other words, a nonmoving mass seems to contain an internal velocity moving forward and backward, a kind of zitterbewegung. It seems that the energy eigenvector represents the physical state, asl the entire model is based on the energy momentum equation (1). The energy matrix does not commute with the velocity operator, meaning that a velocity vector is not a stable physical state. The physical state is a combination of the two velocity eigenvectors, meaning that there is acceleration occurring. Consider a single velocity vector (say with eigenvalue +1). The system then can operate on this with the energy matrix or H. Using (6), we obtain the -1 eigenvector. Operating again, the +1 vector. Thus, H is acting like a time evolution operator, just as in quantum mechanical formalism. In other words, quantum mechanical formalism is appearing directly from a matrix formalism of the classical equation (1). In addition, a time evolution of exp (-iHt) also follows. For the case here of p=0, we can write: cos(wt) (velocity eigenvector +1) - i sin(wt) (7) (velocity eigenvector -1) with w=m. To see this another way, consider that one is ¨jumping¨ from the + velocity vector to the -velocity vector in time, back and forth. Such a motion, leading to equal weights is given by (7). Thus, it appears that dO/dt = [H, O] also appear just as in quantum mechanics, because these follows from the evolution of the operator: O(t)=exp(iHt) O exp(-iHt)

For the case of p not equal to 0, the same time evolution, namely exp (-iHt), also holds. Consider:
exp(-iHt) (1,0) i.e. the evolution of one of positive velocity eigenvector.

H (1,0) = (m, -p) so exp(-iHt) (1,0) = cos(Et) (0,1) - i sin(Et) (m/E, -p/E)
The average of the square of cos(Et) and sin(Et) are the same, so the root mean square vector is: (0,1) + (m/E, -p/E) which is proportional to (m, E-p) which is the positive energy eigenvector.

Thus, the positive energy eigenvector can be thought of in terms of a kind of oscillation between two other type of vectors (here a velocity eigenvector and (m/E, -p/E). It is possible that there is a physical reality to this oscillation, even though looking only at the energy eigenvector (m, E-p) one sees no such oscillation.

It seems further that one can repeat the exact calculations given in [1] for quantum zitterbewegung with the energy matrix H and the velocity operator used here, even though we have assumed no quantum mechanical methods.
We have v= |1 0 | for velocity and
| 0 -1 |

[H, velocity operator] = 2m | 0 -1 | or 2{ p | 1 0| - (v operator) H }
|1 0 | | 0 1|

which is the same result given in [1].

One can also make a momentum operator using (velocity matrix) x |p m | = | p -m|
| m -p| | m p |
The opposite order gives the transpose, although the two do not commute. The eigenvalues in each case are p+im and p-im. The expectation value is p in both cases and the contributions of the two components of the eigenvector are similar to the velocity contributions. The momentum

operators do not commute with the Energy matrix |p m|
|m -p|

For a particle at rest, the momentum operator is |0 -m| and the expectation with the energy
| m 0|

eigenvector is 0.

As mentioned, the two dimensional model for +E and -E gives 2-vectors with sqrt(1+v^2) and sqrt (1 - v^2) components suggesting a kind of internal frequency for the particles, because both the +1 and -1 eigenvalues of the velocity operator are part of the energy eigenvectors . The +E has one kind of frequency and the -E the opposite. This may be why special relativity can map into both spin up and down and particle and antiparticle.

De Broglie had already postulated that the electron has an internal clock, but the above simple model may show more clearly the origins of such a possible frequency. If one considers a specific frequency, it follows that there is a characteristic length associated with the corresponding period. This may describe the idea of the wavelength.

Conclusion

The Dirac equation is a quantum mechanical equation.The zitterbewegung argument of Schrodinger is based on the Dirac equation and starts with a quantum mechanical derivation of the velocity operator. This gives a sense that one needs a theory of quantum mechanics in order to have a velocity operator and zitterbewegung. The Barut Zanghi equations for the electron are classical equations which exhibit classical zitterbewegung. The main argument of this note, which uses a simple one-dimensional model and linear algebra, is that a velocity operator, which does not commute with the energy matrix H, and zitterbewegung are contained directly in (1), without any assumptions of quantum mechanics. It appears that (1) contains both a particle with momentum p and mass m, in addition to some ¨inner¨ motion with a frequency described by a velocity operator. One can go further to show that exp(-iHt) is the time evolution operator in this model, from which it follows that dO/dt = [H, O]. One can obtain the zitterbewegung equations of the same form as the quantum mechanical case given in [1]. These ideas seem to point to dual particle wave behaviour (in that there is p for the particle plus a periodic internal motion) and a quantum mechanical formalism directly contained in Einstein´s 1905 equation (1), if one accepts a priori the existence of +E and -E. At the time, there was probably reluctance to accept any model with -E. As far as we know, this idea has not already been presented in the literature. Barut and Zhangi also derive a classical zitterbewegung, but do so using a postulated Lagrangian.

References

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