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BANACH CONTRACTIVE THEOREM ON COMPLETE METRIC SPACE –A NEW APPROACH

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Abstract: In this paper we investigate some properties of fuzzy metric space and we give fixed point theorem for complete metric space. We extend the fixed point theorem for fuzzy contractive mappings on complete fuzzy metric space. And considered fuzzy version of the classic Banach contractive theorem on complete space.

Key words: Fuzzy sets, metric space, fuzzy contractive mapping, complete metric space, contractive sequence,

1. Introduction:

The concepts of fuzzy set were first introduced by Zadeh.[9] Chang used the idea of fuzzy set to introduce fuzzy topological space. Many authors have introduced the concept of fuzzy metric in different ways. George and Veeramani [6] modified the concepts of fuzzy metric space introduced by Kramosil and Michalek [7]. In particular they obtained Hausdorff topology for this kind of fuzzy metric spaces. Recently it was proved that the topology induced by a fuzzy metric space. And they were given a necessary and sufficient condition for a fuzzy metric space to be complete. Fuzzy metric space to be compacts. Fixed point theory for contractive type mapping in fuzzy metric spaces is closely related to the fixed point theory to the same type of mappings in probabilistic metric space of menger type .The purpose of this article is investigated. Some properties of fuzzy metric space and we give fixed point theorem for complete metric space. We extend the fixed point theorem for fuzzy contractive mappings on complete fuzzy metric space. And considered fuzzy version of the classic Banach contractive theorem on complete space. In this paper we investigate some properties of fuzzy metric space and we give fixed point theorem for complete metric space. We extend the fixed point theorem for fuzzy contractive mappings on complete fuzzy metric space. And considered fuzzy version of the classic Banach contractive theorem on complete space.

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2. PRELIMINARIES

In this section we define basic concepts of metric spaces.

Definition 2.1: A binary operator $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norms if $*$ satisfied the following conditions

(i) $*$ is associative and commutative

(ii) $*$ is continuous

(iii) $* 1 = a$,for every $a \in [0,1]$

(iv) $a * b \leq c * d$ Where $a \leq b$ and $c \leq d$ and $a, b, c, d \in [0,1]$

Definition 2.2: The 3 tuple $(X, Q,*)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t- norms and Q is a fuzzy set on $X \times [0, \infty)$ satisfying the following conditions

(i) $Q(x, y, 0) > 0$

(ii) $Q(x, y, t)$ if and only if x = y

(iii) $Q(x, y, t) = Q(y, x, t)$

(iv) $Q(x, y, t) * Q(x, z, s) \leq Q(x, z, t + s)$

(v) $Q(x, y, .): (0, \infty) \to [0,1]$ is continuous for $x, y, z \in X$ and $t, s > 0$

Definition 2.3: Let $(X, Q,*)$ is said to be a fuzzy metric space. The open ball $B(x, r, t)$ for $t > 0$ with center $x \in X$ and radius r, 0<r<1 is defined as $B(x, r, t) = \{y \in X / Q(x, y, t) > 1 - r\}$ the family $\{ B(x, r, t) / x \in X, 0 < r < 1, t > 0\}$ is a neighbourhood's system for Hausdorff topology on X that we call induced by the fuzzy metric space Q.

Definition 2.4: In metric space (X, d) the 3 tuple $(X, Q,*)$ where $Q_d(x, y, t) = \frac{t}{t+d(x,y)}$ where $a * b = ab$ is a fuzzy metric space. This M_d is called the standard fuzzy metric induced by "d".

Proposition 2.5: In fuzzy metric space $(X, Q,*)$ for any $r \in [0,1]$ we can find an $s \in (0,1)$ such that $s * s \geq r$

Definition 2.6: Let $(X, Q,*)$ be a fuzzy metric space. We will say the the mapping $h: X \to X$ is fuzzy contractive if there exists $k \in (0,1)$ such that $\frac{1}{Q(h(x,h(y),t)} - 1 \leq k\left(\frac{1}{Q(x,y,t)} - 1\right)$ for each $x, y \in X$ and $t > 0$ is called the contractive constant of 'h'.

The above definition is justified by the next proposition -1

Proposition 2.7: Let (X, d) be a metric space the mapping $h: X \to X$ isa contractive a contraction on the metric space (X, d) with contractive constant K if and only if h is fuzzy contractive with contractive constant K on the standard fuzzy metric space $(X, Q_d,*)$ induce by

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d. Recall that a sequence $\langle x_n \rangle$ a metric space (X, d) is said to be contractive if there exists a $k \in (0,1)$ such that $d(x_{n+1}, x_{n+2}) \leq k(d_{xn}, x_{(n+1)})$ for all $\in N$.

Now we propose the following definitions compare with definition 2.6

Definition 2.8: Let $(X, Q,*)$ be a fuzzy metric space we will say that the sequence $\langle x_n \rangle$in X is fuzzy contractive if there exists $k \in (0,1)$ such that $\frac{1}{Q(x_{n+1}, x_{n+2}, t)} - 1 \leq k\left(\frac{1}{Q(x_{n+1}, x_{n+2}, t)} - 1\right)$ for all $t > 0, n \in N$.

Proposition 2.9: Let $(X, Q,*)$ be the standard fuzzy metric space induced by the metric d on X. The sequence $\langle x_n \rangle$ in X is contractive in (X, d) if and only if $\langle x_n \rangle$ is fuzzy contractive in$\langle X, Q,* \rangle$.

Definition 2.10: A sequence $\langle x_n \rangle$ is fuzzy metric space $(X, Q,*)$ is a Cauchy sequence if and only if for each $\delta \in (0,1)$ and $t > 0$there exits $n_0 \in N$ such that $Q(x_n, x_m, t) \geq 1 - \delta$ for $n, m \geq n_0$.

Definition 2.11: A fuzzy metric space is said to be complete if every Cauchy sequence is convergent.

3. STANDADED RESULT

In this section we will study the characterization of fuzzy metric space and we give fixed point theorem for complete fuzzy metric space.

Theorem 3.1: Subspace of separable fuzzy metric space is separable.

Proof:

Let X be given fuzzy metric space and Y be subspace of X. Let $P = \{x_n / n \in N\}$ be a countable dense subset of X. For arbitrary but fixed $n, k \in N$ if there are points $x \in X$ such that $Q\left(x_n, x, \frac{1}{k}\right) > 1 - \frac{1}{k}$ choose one of them and dense it by x_{nk}. Let $B = \{x_{nk} / n, k \in N\}$ then B is countable.

Now we show that $Y \subset \bar{B}$. Let $y \in Y$ given $r, t > 0, 0 < r < 1$, we can find a $k \in N$ such that $\left(1 - \frac{1}{k}\right) * \left(1 - \frac{1}{k}\right) > 1 - r$. Since P is dense in X there exists an $m \in N$ such that $Q\left(x_m, y, \frac{1}{k}\right) > 1 - \frac{1}{k}$ But by definition of B, there exists $x_{nk} \in P$ such that $Q\left(x_{mk}, x_m, \frac{1}{k}\right) > 1 - \frac{1}{k}$

Now

$$Q(x_{mk}, x_m, t) \geq Q\left(x_{mk}, x_m, \frac{t}{2}\right) * \left(x_m, y, \frac{t}{2}\right)$$

$$\geq Q\left(x_{mk}, x_m, \frac{1}{k}\right) * \left(x_m, y, \frac{1}{k}\right)$$

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$$\geq \left(1-\frac{1}{k}\right) * \left(1-\frac{1}{k}\right)$$
$$> 1-r$$

Thus $Y \in \bar{B}$ and hence Y is separable.

Theorem 3.2 [4]: A sequence $\langle x_n \rangle$ in a metric space $(X, Q, *)$ converges to x if and only if $Q(x_n, x, t) \to t$ as $n \to \infty$

Proposition 3.3: A convergent sequence is Cauchy.

Proof: It is easy to check by proposition 5

Lemma 3.4 [4]: The metric space (X,d) is complete if and only if the standard fuzzy metric space $(X, Q, *)$ is complete.

Next we extended the fixed point theorem to fuzzy contractive mapping of complete fuzzy metric space.

Theorem 3.5: Let $(X, Q, *)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let $h: X \to X$ be a fuzzy contractive mapping being k the contractive constant then h has unique fixed point.

Proof:

Fix $x \in X$

Let $x_n = h(x), n \in N$, we have for $t > 0$

$$\frac{1}{\mathrm{Q}(h(x), h^2(x), t)} - 1 \leq k\left(\frac{1}{Q(x, x, t)} - 1\right)$$

and by induction

$$\frac{1}{\mathrm{Q}(x_{n+1}, x_{n+2}, t)} - 1 \leq k\left(\frac{1}{Q(x_{n+1}, x_{n+2}, t)} - 1\right), n \in N$$

Then $\langle x_n \rangle$ is a fuzzy contractive sequence so it is a cauchy sequence and hence $\langle x_n \rangle$ convergence to y for some $y \in X$. We will see y is a fixed point of h, by theorem 1, we have

$\frac{1}{\mathrm{Q}(h(y), h(x), t)} - 1 \leq k\left(\frac{1}{Q(y, x_n, t)} - 1\right) \to 0$ as $n \to \infty$

Then

$\lim_{n\to\infty} Q(h(y), h(x_n), t) = 1$ for each $t > 0$ and therefore $\lim_{n\to\infty} h(x) = h(y)$

(ie) $\lim_{n\to\infty} x_{n+1} = h(y)$ and then $h(y) = y$

To show uniqueness assume $h(x) = z$

We have

$$\frac{1}{\mathrm{Q}(y, z, t)} - 1 = \left(\frac{1}{Q(h(y), h(z), t)} - 1\right)$$

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$$\leq k\left(\frac{1}{Q(y,z,t)} - 1\right)$$

$$=k\left(\frac{1}{Q(h(y),h(z),t)} - 1\right)$$

$$\leq k^2\left(\frac{1}{Q(y,z,t)} - 1\right)$$

$$\leq \cdots$$

$$\leq k^n\left(\frac{1}{Q(y,z,t)} - 1\right) \to 0 \text{ as } n \to \infty$$

Hence $Q(y,z,t) = 1$ and then $y = z$

Now suppose that $(X, Q,*)$ is a complete standard fuzzy metric space induced by the metric "d" on X.

From Lemma 3.4 (X, d) is complete then $\langle x_n \rangle$ is a fuzzy contractive sequence by proposition 2.9 it is contractive in (X, d) hence convergent. So form Proposition 3.3, we have the following corollary which can be dealt the fuzzy version of the classic contractive theorem on complete metric space.

Corollary – 3.6: Let$(X, Q,*)$ be a complete standared fuzzy metric space and let $h: X \to X$ is a fuzzy contractive mapping then "h" has unique fixed point.

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