Global existence solutions of Fuzzy Cauchy′s Problem of Second order differential equations under Hausdorff distance

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Abstract: In this paper, we study the numerical method for solving second order Fuzzy Cauchy's problem with fuzzy initial condition .We propose a new method of how local and global existence solution can be constructed. Finally, We present an example with initial condition having four different solution.

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Section 1 Introduction:

Fuzzy differential equation have been applied extensively in recent years to model uncertainty in mathematical models. First order ordinary differential equation, In particular fuzzy cauchy's differential equations are one of the simplest fuzzy differential equations which may appear in many applications. FDE are important tools to deal with dynamic system if fuzzy environment. The concept of a fuzzy derivative was first introduced by s.L.Chang and L.A.Zadah [7]. Fuzzy differential equations and IVB were regularly treated by O.Kaleva [11]. Finding a solution to FDE which satisfies necessary local and global existence conditions in our main goal in this paper. Since it is too complicated to find an exact solution, numerical methods are more suitable for mentioned problem. The topic of FDE's has been rapidly growing in recent years. Dubois and Prade [8,9] who defined and used the extension principle Kendal and Byatt [13] applied the concept FDE to the analysis of Fuzzy dynamic problems. Puri and Ralescu defined the concept H differentiability and Seikkala[17] generalized FDE's. Kaleva, Seikkala,[16]. He and Yi, Kloeden, Menda and finally Friedman, Ma and Kendal, concentrate on Fuzzy Cauchy's Problem. The main directions of development of subject have been diverse with application to varity of real problems such as quantum topics, gravity, Chaotic system medicine and engineering problems. In this paper we focus on the cauchy's problem for ordinary differential equations of second order with initial conditions. Using Hausdorff distance a numerical procedure in presented, In section 2 first we briefly introduced preliminary topics such as Fuzzy number, Hausdorff distance, convex set, support of fuzzy set and then a fuzzy cauchy's problem is defined. In section 3 a new algorithm for Fuzzy cauchy's problem of second order is mentioned. In section 4 some numerical examples are given with illustration.

Section 2: Preliminaries

In this section, We consider the first order fuzzy differential equation y'= f (t , y) where y is a function of t, y (t , y) is a fuzzy function of crisp variable t and fuzzy variable y, and y' is the fuzzy derivative of y. If an initial value Y (ε_0) = $y_0 \in E^n$ is gives, we obtain a fuzzy Cauchy problem of first order.

Definition 2.1: [**Fuzzy set**]: A fuzzy set $\hat{A}$ is defined by $\tilde{A} = \{(x, \mu_A(x); x \in A, \mu_A(x) \in [0,1])\}$.

In the pair (x, $\mu_A(x)$), the first element x belong to the classical set A, the second element $\mu_A(x)$, belong to the interval [0,1], called Membership function.

Definition 2.2: [**Support of Fuzzy set**]: The Support of fuzzy set $\widetilde{A}$ is the set of all points x in X such that $\mu_A(x)$>0. That is Support ($\widetilde{A}$) = $\{x/\ \mu_A(x) > 0\}$.

Definition 2.3: [$\alpha - cut$:]:The $\alpha - cut$ of $\alpha - level$ set of fuzzy set $\widetilde{A}$ is a set consisting of of those elements of the universe X whose membership values exceed the threshold level α.

That is $\tilde{A}_{\alpha} = \{x/\mu_{\tilde{A}}(x) \geq \alpha\}$.

Definition 2.4:[Convex]: A fuzzy set $\hat{A}$ is convex if $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), (\mu_{\tilde{A}}(x_2))$, $x_1, x_2 \in$ X and $\lambda \in [0,1]$. Alternatively, a fuzzy set is Convex , if all $\alpha - level$ sets are convex.

Definition 2.5: [Fuzzy Number]: A Fuzzy set $\hat{A}$ on R must possess at least the following three properties to qualify as a fuzzy number.

(i) $\tilde{A}$ must be a normal fuzzy set.

(ii) $\alpha_{\tilde{A}}$ must be closed interval for every $\alpha \in [0,1]$

(iii) the Support of $\hat{A}$,$0+_{\tilde{A}}$, must be bounded.

Definition 2.6: Let X and Y be two non-empty subsets of a metric space (M, d). We define their Hausdorff distance $d_H(X,Y)$ by

$D_H(X,Y) = max\left\{\sup_{x \in X, y \in Y} \inf d(x,y), \sup_{y \in Y, x \in X} \inf d(x,y),\right\}$ where sup represents the supremum and inf the infimum.

Definition 2.7: A triangular fuzzy number is defined as a fuzzy set in E^x which is specified by an ordered triple $u_1, u_2, u_{3)} \in R^3$ with $u_1 \leq u_2 \leq u_3$ such that $[u]^0 = [u_1, u_3]$ *and* $[u]^1 = \{u_2\}$ *there* for $0 \leq \alpha \leq 1$

We have $[u]^\alpha = [u_2 - (1-\alpha)(u_2 - u_1), u_2 + (1-\alpha)(u_3 - u_2)]$

For u, v ∈ R_f and K ∈ R, the sum u + v and product are defined by Ku are defines by $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$, $[Kv] = k[v]^\alpha$ for any αt[0, 1], where $[u]^\alpha + [v]^\alpha = \{ x + y/x \in [u]^\alpha, y \in [v]^\alpha \}$ means the usual addition of two intervals of R and $k[u]^\alpha = \{Kx/x \in [u]^\alpha\}$ mean the usual product between a scalar and a subset of R, let,

D : R_f x R_f → R + U{0},

D : (u, v) = $\underset{\alpha t[0,1]}{Sup}$ max { $|u_-^\alpha - v_+^\alpha|, |u_-^\alpha, u_+^\alpha|$ }

Be the Hausdorff distance between two fuzzy no's when [u] α = $[u_-^\alpha, u_+^\alpha]$, [v] α = $[v_-^\alpha, v_+^\alpha]$. the following properties are well known [17]

(a) D (u + v, v + w) = D(u, v), for all u, v, w ∈ R_f.

(b) D(ku, kv) = |k| d(u, v), for all k∈R, u, v ∈R_f

(c) D(u + v, w + e) = D(u, w) + D(v, e) , for all u, v, w, e ∈ R_f

and (R_f, D) is a complete metric space.

Section -3 Algorithm for fuzzy Cauchy's problem of second order

In this section, we study the Cauchy's problem for the second order differential equations

$y''(t) + a(t)y'(t) + b(t)y(t) = w(t)$ ⟶ (1) $y(0) = c_1$, $y' = c_2$ where $C_1, C_2 \in R_f$ and a(t), b(t) > 0, our strategy of solving (1) is based on the selection of derivatives type is the Cauchy's problem equation. We first give the following definition for the solutions of (1)

Let y:[a, b] → R_f be a fuzzy valued function and n, m = 1,2 on [A,B]. One says y is an (n, m) solution for problem (1). If $D_n^{(1)}$ y(t), $D_{n,m}^{(2)}$ y(t) exist and $D_{n,m}^{(2)} y(t) + a(t) D_n^{(1)} y(t) + b(t)y(t) = w(t)$, $y(0) = c_1$, $D_n^{(1)} y(0) = c_2$.

Therefore, four ordinary differential equations systems are possible for (1) as follows:

(1, 1) System

$$\left.\begin{array}{c} y_-''(t;\alpha) + a(t)y_-'(t;\alpha) + b(t)y_-(t;\alpha) = w(t) \\ y_+''(t;\alpha) + a(t)y_+'(t;\alpha) + b(t)y_+(t;\alpha) = w(t) \\ y_-(0;\alpha) = c_{1-}^\alpha \; ; \; y_+(0;\alpha) = c_{1+}^\alpha \\ y' - (0;\alpha) = c_{2-}^\alpha \; ; \; y_+'(0;\alpha) = c_{2+}^\alpha \end{array}\right\} \longrightarrow (2)$$

(1, 2) system

$$\left.\begin{array}{c} y_+''(t;\alpha) + a(t)y_-'(t;\alpha) + b(t)y_-(t;\alpha) = w(t) \\ y_-''(t;\alpha) + a(t)y_+'(t;\alpha) + b(t)y_+(t;\alpha) = w(t) \\ y_-(0;\alpha) = c_{1-}^{\alpha} \; ; \; y_+(0;\alpha) = c_{1+}^{\alpha} \\ y_-'(0;\alpha) = c_{2-}^{\alpha} \; ; \; y_+'(0;\alpha) = c_{2+}^{\alpha} \end{array}\right\} \longrightarrow (3)$$

(2, 1) system

$$\left.\begin{array}{c} y_+''(t;\alpha) + a(t)y_+'(t;\alpha) + b(t)y_-(t;\alpha) = w(t) \\ y_-''(t;\alpha) + a(t)\, y_-'(t;\alpha) + b(t)y_+(t;\alpha) = w(t) \\ y_-(0;\alpha) = c_{1-}^{\alpha} \; ; \; y_+(0;\alpha) = c_{1+}^{\alpha} \\ y_-'(0;\alpha) = c_{2-}^{\alpha} \; ; \; y_+'(0;\alpha) = c_{2+}^{\alpha} \end{array}\right\} \longrightarrow (4)$$

(2, 2) – system

$$\left.\begin{array}{c} y_-''(t;\alpha) + a(t)\, y_+'(t;\alpha) + b(t)y_-(t;\alpha) = w(t) \\ y_+''(t;\alpha) + a(t)\, y_-'(t;\alpha) + b(t)y_+(t;\alpha) = w(t) \\ y_-(0;\alpha) = c_{1-}^{\alpha} \; ; \; y_+(0;\alpha) = c_{1+}^{\alpha} \\ y_-'(0;\alpha) = c_{2-}^{\alpha} \; ; \; y_+'(0;\alpha) = c_{2+}^{\alpha} \end{array}\right\} \longrightarrow (5)$$

The following example is based on the above algorithm.

Section-4 Numerical Example

Example-4.1 : In the Fuzzy real line R_L, We define a map $d: R_L \to [0,1]$ called standard metric and Hausdorff metric is defined by

$D_n = \frac{t_\alpha}{t_\alpha + d(x,y)}$, d(x,y) = min { |f(x) + f(y), f(x) – f(y)|} x,y in R_L

Using this, we got so many image values for $\forall\, x, y \in R_L$ different Cauchy's solutions.

Regarding (1,1) solution $t_\alpha = 0.1$ *and* $f(x) = 0.2, 0.4, 0.5;\ f(y) = 0.3, 0.4, 0.6$

$$D_1 = 0.5; D_2 = 1; D_3 = 0.5$$

Similarly For the same $t_\alpha = 0.1$,

$(1,2)$ solution is obtained by $D_1 = 0.526, D_2 = 0.625, D_3 = 0.625$ into $f(x) = \frac{1}{4}, \frac{1}{5}, \frac{1}{7}$ *and* $f(y) = \frac{1}{6}, \frac{1}{7}, \frac{1}{5}$

For (2,1) solution by $D_1 = 0.333; D_2 = 0.153; D_3 = 0.476$ *with* $f(x) = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ *and* $f(y) = 0.5, 0.75, 0.25$

Finally (2,2) solution by $D_1 = 0.333; D_2 = 0.333; D_3 = 0.333$ *along with* $f(x) = 0.1,\ 0.2, 0.3;$ And $f(y) = 0.3, 0.4, 0.5.$ So that Various images are got from the Cauchy's problem.

2. $y'' = g(t, y(t)), y(t_0) = y_0 \;\; \to (1)$

sufficient conditions for existence of a unique solution to equation (1) are

(i) Continuity of f

(ii) Lipschitz condition which deckus

$D(g_1(t, y), g_2(t, y)) \leq L\, D(x, y)$ for some $L > 0$.

Now, for y(t) to be a solution of fuzzy Cauchy's problem we need that y′ exists but also equation (1) must hold. To check equation (1), first we have to compute g(t,y). α- levels of f(t,y) can be found as follows $(g(t,y))^{\alpha} = [g_1(t,\alpha), g_2(t,\alpha)]$ with

$g_1(t, \alpha) = \min \{ g(t, y) / y \in y(t)^{\alpha} \}$

$g_2(t, \alpha) = \max \{ g(t, y) / y \in y(t)^{\alpha} \}$ for $t \in I$, $\alpha \in [0,1]$

we say that y is a solution of (1), if y′ exists and

$y_1''(t;\alpha) = g_1(t, \alpha)$, $y_1(t_0;\alpha) = = y_0^1(\alpha) \rightarrow$ (2)

$y_2''(t;\alpha) = g_2(t, \alpha)$, $y_2(t_0;\alpha) = = y_0^2(\alpha) \rightarrow$ (3) where $y(t_0;\alpha) = [y_0^1(\alpha), y_0^2(\alpha)]$

Example-4.2: Let us consider the following second order fuzzy Cauchy problem

$y''(t) + y'(t) + y(t) = t$, $y(0) = 0$, $y'(0) = 1$,

Where 0 and 1 are triangular fuzzy number having α-level sets $[0]^{\alpha} = [\alpha-1, 1-\alpha]$ and $[1]^{\alpha} = [\alpha, 2 - \alpha]$, $\alpha \in [0, 1]$ respectively.

We obtain,

(1, 1) solution

$y_-(t;\alpha) = (\alpha - 1) + \alpha t + ((1 - 2\alpha)/2)t^2 + (1/6)t^3$

$y_+(t;\alpha) = (1 - \alpha) + (2 - \alpha)t + ((2\alpha - 3)/2)t^2 + (1/6)t^3$, $t \in [0, +\infty]$

(1, 2) solution

$y_-(t;\alpha) = (\alpha - 1) + \alpha t + ((2\alpha - 3)/2)t^3 + (1/6)t^3$

$y_+(t;\alpha) = (1 - \alpha) + (2 - \alpha)t + ((1 - 2\alpha)/2)t^2 + (1/6)t^3$, $t \in [0, +\infty]$

(2, 1) solution

$y_-(t;\alpha) = (\alpha - 1) + \alpha t - (1/2)t^2 + (1/6)t^3$

$y_+(t;\alpha) = (1 - \alpha) + (2 - \alpha)t + ((1 - 2\alpha)/2)t^2 + (1/6)t^3$, $t \in [0, +\infty]$

(2, 2) solution

$y_-(t;\alpha) = (\alpha - 1)t + \alpha t - (1/2)t^2 + (1/6)t^3$

$y_+(t;\alpha) = (1 - \alpha)t + (2 - \alpha)t - (1/2)t^2 + (1/6)t^3$, $t \in [0, +\infty]$

Example-4.3: Consider the second order fuzzy Cauchy's problem $y''(t) = w_0$, $y(0) = c_1$, $y(0) = c_2$, $t \geq 0$.

Where $W_0 = C_1 = C_2 = [\alpha - 1, 1 - \alpha]$. It posses four different solutions. In order to extend the results to n^{th} order fuzzy Cauchy's problems. We have 2^n solutions for a nth order equation by using the different types of derivatives.

Conclusion: The obtained solution in this method is equivalent or closed to the exact solution of the problem. The solution of fuzzy differential equation under generalized differentiability can be obtained by Hausdorff distance. In feature, Hausdorff distance can be used to solve Non – linear stochastic differential equation in the Fuzzy environment.

References

[1] S. Abbasbandy, T. Allahviranloo, Numerical solutions of fuzzy differential equations by Taylor method, Journal of Computer and Mathematics with Applications 2 (2002) 113-124

[2] T. Allahviranloo, N. Ahmadi, E. Ahmadi, Numerical solution of fuzzy differential equations by predictor-corrector method, Information Sciences 177 (2007) 1633-1647.

[3] E. Babolian , H. Sadeghi b, Sh. Javadi. Numerically solution of fuzzy differential equations by Adomian method, Applied Mathematics and Computation 149 (2004) 547-557.

[4] B. Bede, S.G. Gal, Generalizations of the differentiability of fuzzy number value functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151 (2005) 581-599.

[5] B. Bede, I.J. Rudas, A.L. Bencsik, First order linear fuzzy differential equations under generalized differentiability, Information Sciences 177 (2007)1648-1662.

[6] J.J. Buckley, T. Feuring, Fuzzy differential equations, Fuzzy Sets and Systems 110 (2000) 43-54.

[7] S. L. Chang, L. A. Zadeh, On fuzzy mapping and control, IEEE Transactions on Systems Man Cybernetics 2 (1972) 330-340.

[8] D. Dubois, H. Prade, Towards fuzzy differential calculus: Part 3, Differentiation. Fuzzy Sets and Systems 8 (1982) 225-233.

[9] D. Dubois, H. Prade, Fuzzy numbers: an overview, in: J. Bezdek (Ed.), Analysis of Fuzzy Information, CRC Press (1987) 112-148.

[10] M. Friedman, M. Ma, A. Kandel, Numerical solution of fuzzy differential and integral equations, Fuzzy Sets and Systems 106 (1999) 35-48.

[11] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301-317.

[12] A. Kandel, W. J. Byatt, Fuzzy differential equations, in: Proceedings of International Conference Cybernetics and Society, Tokyo: (1978) 1213-1216.

[13] A. Kandel, W. J. Byatt, Fuzzy processes, Fuzzy Sets and Systems 4 (1980)117-152.

[14] A. Khastan, K. Ivaz, Numerical solution of fuzzy differential equations by Nystrom method. Chaos Solitons Fractals 41 (2009) 859-868.

[15] J.J. Nieto, R. Rodrłguez-Lpez, D. Franco, Linear first-order fuzzy differential equation, International Journal of Uncertainty Fuzziness Knowledge-Based Systems 14 (2006) 687-709.

[16] M. Puri, D. Ralescu. Differentials of fuzzy functions.Journal of Mathematical Analysis and Applications 91(1983) 552-558.

[17] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319-330. [18] S. J .Song, C. X. Wu, Existence and uniqueness of solutions to the Cauchy problem of fuzzy differential equations, Fuzzy Sets and Systems 110 (2000) 55-67.

[18] J. P. Xu, Z. Liao, Z. Hu, A class of linear differential dynamical systems with fuzzy initial condition. Fuzzy Sets and Systems 158 (2007) 2339-2358.

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